Faculty of Mathematics and Computer Science
Mathematical Institute

Pin bordism in low degrees of classifying spaces

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by
Thorsten Hertl

First Supervisor:
Prof. Dr. Thomas Schick

Second Supervisor:
Prof. Dr. Tammo tom Dieck

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Introduction

The concepts of bordism and cobordism is over 100 years old. In the 19th century, Henri Poincaré introduced bordism, hoping to define (ordinary) homology purely in terms of manifolds.

In the early 50's of the last century, bordism appeared in homotopy theory. In 1950, Lev Pontrjagin used framed bordism to gain informations about higher homotopy groups. Shortly after that, René Thom used homotopy theory to classify (oriented) compact manifolds up to (oriented) bordism.

Then, in the 60's, Michael Atiyah and Friedrich Hirzebruch studied bordism intensively as a generalised homology theory.

Later, in the Eighties, bordism showed up in modern quantum physics. Indeed, many invariants that arise in quantum field theory turn out to be invariant under a certain bordism relation. Having this in mind, it is not too surprising that Edward Witten asked how

$$\Omega_4^{\mathrm{Pin}^-}(B\mathrm{O}(2))$$

looks like. The answer to this question is the purpose of this thesis.

The text is structured in the following way. The first of three chapters studies the fundamental objects of this thesis. After a brief recapture of the (S)Pin groups, we discuss Pin structures on manifolds. In particular, we will see that Pin structures descend from manifolds to their boundaries in a unique way, allowing the definitions of Pin bordism groups. We will also give a reduction to Spin bordism and calculate the needed Pin bordism coefficient groups.

The second chapter provides information about the topological space $B\mathrm{O}(2)$. We will give a cell decomposition and use that to determine the homology groups of $B\mathrm{O}(2)$. In addition, we will determine its unoriented bordism group $\Omega_*^{\mathrm{O}}(B\mathrm{O}(2))$.

In the third and final chapter, we will present the answer to the motivating question. In sections 3.2 and 3.3, we will show that

$$\Omega_4^{\mathrm{Pin}^-}(B\mathrm{O}(2)) \cong \mathbb{Z}_2^2.$$

Additionally, we will calculate all other Pin bordism groups up to degree 4. We will do this not only for $B\mathrm{O}(2)$, but also for $B\mathrm{SO}(2)$ and $B\mathbb{Z}_2$, and provide geometric representatives of the generators.

Chapter 1

The Pin groups and its bordism

The aim of this chapter is to give a comprehensive introduction into the essential concepts of this thesis, most notably Pin structures and bordism groups. Since the Pin groups are not as famous as their little brothers, the Spin groups, we will review their construction in the first section and elementary properties will be derived. In section 1.2 we will study how and when they arise on bundles. Although this does seem to be very spectacular, the most important result of this section is possibly Theorem 1.2.17 which says that there is a canonical bijection between Pin structures and stable Pin structures. This allows us to define the Pin bordism groups in section 1.3. Since Pin and Spin are closely related, one should expect that their bordism theories are closely related and indeed, Pin bordism can be completely described in term of Spin bordism. This reduction is presented in section 1.4. Finally, in section 1.5 we calculate the Pin bordism coefficient groups of our interest with the help of the Atiyah-Hirzebruch spectral sequence.

1.1. Construction of the Pin groups

The purpose of this section is to give a short reminder of the theory of Pin and Spin groups and to fix some notation. We will only cover the information we need for this thesis, so the material presented here is only the tip of the iceberg. For further information see [LM89]. Nevertheless, the presentation of the material will be (nearly) self contained. We construct the Pin and Spin using covering theory and classify all Spin and Pin groups up to isomorphism of topological groups. Then, a concrete model using Clifford algebras will be provided, which we will later use for calculations. We assume that the reader is familiar with the foundations of covering theory on a level presented in [SZ94, Chapter 5]. In particular, the lifting theorem for coverings should be known.

Theorem 1.1.1. *Let G be a path connected topological group and $p\colon (\tilde{G}, 1) \to (G, 1)$ be a covering with path connected total space. Then there is a unique group structure on $\tilde{G}$ with 1 as neutral element such that p becomes a group homomorphism. Furthermore, two equivalent covers are also isomorphic as groups.*

Proof. Since every topological group is an H-space, its fundamental group will be commutative. Let us denote the continuous group multiplication of

G with μ. Under the isomorphism $\pi_1(G \times G) \cong \pi_1(G) \oplus \pi_1(G)$ the induced map μ_* corresponds to the addition on $\pi_1(G)$. In other words,

$$\mu_*([f \times g]) = [f] + [g].$$

This implies

$$\mathrm{im}\,(\mu \circ p \times p)_* = \mathrm{im}\, p_* + \mathrm{im}\, p_* = \mathrm{im}\, p_*.$$

Therefore, we obtain a unique lift for $\mu \circ p \times p$, i.e. a continuous map $\tilde{\mu}$: $\tilde{G} \times \tilde{G} \to \tilde{G}$ sending $(1, 1)$ to 1 and making the diagram

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \stackrel{\tilde{\mu}}{\dashrightarrow} & \tilde{G} \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ G & \xrightarrow{\mu} & G \end{array}$$

commutative. We need to verify that this lift satisfies the axioms of a group multiplication. For associativity, observe that $\tilde{\mu} \circ (\mathrm{id} \times \tilde{\mu})$ and $\tilde{\mu} \circ (\tilde{\mu} \times \mathrm{id})$ are lifts of $\mu \circ (\mathrm{id} \times \mu) = \mu \circ (\mu \times \mathrm{id})$ sending $(1, 1, 1) \in \tilde{G}^3$ to $1 \in \tilde{G}$. By uniqueness, these maps must be equal.

To see that $1 \in \tilde{G}$ is a left unit, observe that $g \mapsto \tilde{\mu}(1, g)$ covers id_G, which can be rewritten as $g \mapsto \mu(1, g)$, and sends $1 \in \tilde{G}$ to 1. By uniqueness, $g \mapsto \tilde{\mu}(1, g)$ is the identity on $\tilde{G}$ and therefore 1 a left unit.

The construction of the (left) inverse map runs as follows. Consider the inverse map $\iota\colon G \to G$, $g \mapsto g^{-1}$. On the fundamental group, this map induces $\iota_* = -\mathrm{id}_{\pi_1(G)}$. Therefore, $\mathrm{im}\,(i \circ p)_* = \mathrm{im}\, p_*$, and $\iota \circ p$ lifts to a unique map $\tilde{\iota}\colon \tilde{G} \to \tilde{G}$ satisfying $\tilde{\iota}(1) = 1$. Now, the map $g \mapsto \tilde{\mu}(\tilde{\iota}(g), g)$ covers $g \mapsto \mu(\iota(g), g) = 1$ and sends 1 to 1. Therefore, it must be the constant map sending everything to 1, and this implies that $\tilde{\iota}$ sends every element to its left inverse.

Now, for $i \in \{1, 2\}$, let $p_i\colon (\tilde{G}_i, 1) \to (G, 1)$ be two equivalent coverings and $\varphi\colon G_1 \to G_2$ be a continuous map covering the identity with $\varphi(1_{G_1}) = 1_{G_2}$. Since $\varphi \circ \mu_1$ and $\mu_2 \circ (\varphi \times \varphi)$ cover μ and satisfy $\varphi \circ \mu_1(1, 1) = 1_{G_2} = \mu_2 \circ (\varphi \times \varphi)$, we conclude by uniqueness that both maps are equal. Therefore, the cover-equivalence φ is a group homomorphism. Since φ is a homeomorphism, it is an isomorphism of topological groups. □

Recall that $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$ and that $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$ for $n \geq 3$, so up to covering equivalence there is only one connected two-sheeted covering of $\mathrm{SO}(n)$. By applying the previous theorem to this covering, we prove the existence of the Spin group, defined in the next theorem.

Theorem 1.1.2 (Existence of the Spin group)**.** *Up to isomorphism of topological groups there is a unique topological group* $\mathrm{Spin}(n)$ *satisfying the following properties.*

1. $\lambda\colon (\mathrm{Spin}(n), 1) \to (\mathrm{SO}(n), 1)$ *is a two-sheeted covering.*
2. λ *is a group homomorphism*
3. $\mathrm{Spin}(n)$ *is connected.* □

Corollary 1.1.3. *If $n \geq 3$, then* $\mathrm{Spin}(n)$ *is simply connected, so it is the universal covering of* $\mathrm{SO}(n)$.

The proof for uniqueness uses that a lift of a given continuous map is uniquely determined by the value of a single point, as long as the total space of the covering is path connected. So if we generalise this construction to $\mathrm{O}(n)$, the total space will be non-connected and we cannot expect to have a unique group structure. Indeed, it will turn out that up to isomorphism there will be two different group structures satisfying the above properties, essentially because a set of four elements can be endowed with two different group structures. The next theorem makes this more precise.

Theorem 1.1.4. *Up to isomorphism of topological groups there exist two topological groups* $\mathrm{Pin}^-(n)$ *and* $\mathrm{Pin}^+(n)$ *satisfying the following properties:*

1. $\lambda\colon \mathrm{Pin}^{\pm}(n) \to \mathrm{O}(n)$ *is a two-sheeted covering.*
2. λ *is a group homomorphism.*
3. $\lambda^{-1}(\mathrm{SO}(n)) = \mathrm{Spin}(n)$.

The groups $\mathrm{Pin}^+(n)$ *and* $\mathrm{Pin}^-(n)$ *can be distinguished by the following property. Let* $r \in \mathrm{O}(n) \setminus \mathrm{SO}(n)$ *be any reflection at some hyperplane*[1]*. Then* $\lambda^{-1}(\{\mathrm{id}, r\}) \subseteq \mathrm{Pin}^-(n)$ *is a subgroup isomorphic to* $\mathbb{Z}_4$ *and* $\lambda^{-1}(\{\mathrm{id}, r\}) \subseteq \mathrm{Pin}^+(n)$ *is isomorphic to* $\mathbb{Z}_2^2$.

Proof. We first discuss the question of existence. As sets we define $\mathrm{Pin}^{\pm}(n)$ to be $\mathrm{Spin}(n) \times \mathbb{Z}_2$ and $\lambda := \lambda_{\mathrm{Spin}} \times \mathrm{id}$. Any r as above defines a group homomorphism $r\colon \mathbb{Z}_2 \to \mathrm{O}(n)$ via $r(-1) = r$, so it can be used to identify $\mathrm{O}(n)$ as a semi-direct product of $\mathrm{SO}(n)$ and $\mathbb{Z}_2$; more precisely $\mathrm{O}(n) \cong \mathrm{SO}(n) \rtimes_{c_r} \mathbb{Z}_2$. Using the lifting theorem for coverings it is easy to see that the conjugation with r is a homomorphism $c_r\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{SO}(n))$, which lifts to a homomorphism $\tilde{c}\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{Spin}(n))$.

We define $\mathrm{Pin}^+(n)$ to be $\mathrm{Spin}(n) \rtimes_f \mathbb{Z}_2$. It follows immediately from the construction that λ is a group homomorphism. The first and third condition are also obviously satisfied.

For $\mathrm{Pin}^-(n)$, we define the group multiplication $\mu_{\mathrm{Pin}}\colon \mathrm{Pin}^- \times \mathrm{Pin}^- \to \mathrm{Pin}^-$ via

$$\begin{aligned}
\mu_{\mathrm{Pin}}((x,1),(y,1)) &= (xy, 1)\\
\mu_{\mathrm{Pin}}((x,-1),(y,1)) &= (xf(-1)(y), -1)\\
\mu_{\mathrm{Pin}}((x,1),(y,-1)) &= (xy, -1)\\
\mu_{\mathrm{Pin}}((x,-1),(y,-1)) &= (-xf(-1)(y), 1).
\end{aligned}$$

This product is associative for the same reason the product of a semi-direct product is associative[2]. One easily checks that the neutral element is given by $(1,1)$ and the inverse element of (x,m) is given by $(m\tilde{c}(m)(x^{-1}), m)$. Since $\mu_{\mathrm{Pin}^-}((x_1,m_1),(x_2,m_2))$ and $\mu_{\mathrm{Pin}^+}((x_1,m_1),(x_2,m_2))$ may only differ by a sign in the first component, the covering map λ is also a group homomorphism for the $\mathrm{Pin}^-(n)$ group structure. As in the Pin^+ case, the first and third

[1] In algebraic terms: $r^2 = 1$ and r has signature $(n-1,1)$.

[2] If you are uncomfortable with this argument Theorem 1.1.10 gives the existence and the proof of uniqueness below shows that the product has to be defined in this way.

condition are trivially satisfied. Furthermore, one sees that $\lambda^{-1}(\{\mathrm{id}, r\}) = \{\pm 1, \pm 1\}$. In the Pin^- case, this set is isomorphic to $\mathbb{Z}_4$ with generator $(1,1)$, while, in the Pin^+ case, this set is isomorphic to $\mathbb{Z}_2^2$ with $(1,1)$ and $(1,-1)$ as generators.

Now we turn to uniqueness. For $i \in \{1,2\}$ let $G_i \xrightarrow{\lambda_i} \mathrm{O}(n)$ be two groups satisfying the three conditions in the statement and $r \in \mathrm{O}(n)$ be any reflection at some hyperplane.

First, observe that the isomorphism class of the subgroup $\lambda_i^{-1}(\{\mathrm{id}, r\})$ does not depend on the choice of r. Indeed, let r_2 be another reflection at a hyperplane. Since r and r_2 are symmetric, orthogonal matrices having the same signature, we can find a $g \in \mathrm{O}(n)$ such that $grg^\top = r_2$. By replacing g with $\bar{g} := g \cdot r$ if necessary, we may even assume that g is an orientation-preserving isometry. The conjugation c_g lifts to an automorphism mapping $\lambda^{-1}(\{\mathrm{id}, r\})$ bijectively to $\lambda^{-1}(\{\mathrm{id}, r_2\})$; therefore, both subgroups are isomorphic.

From the third condition, we conclude that $\mathrm{Spin}(n) \subseteq G_i$ is an open and closed subgroup. Thus, it is a connected component containing the unit element, and therefore normal in G_i. Since $G_i/\mathrm{Spin}(n) \cong \mathbb{Z}_2$, the groups decompose $G_i = \mathrm{Spin}(n) \sqcup \mathrm{Spin}(n) \cdot \tilde{r_i}$, where $\lambda_i(\tilde{r_i}) =: r$ is a reflection at some hyperplane. In other words, any element $x \in G_i$ has a unique decomposition $x = g \cdot \tilde{r_i}$ with $g \in \mathrm{Spin}(n)$.

Define $\varphi\colon G_1 \to G_2$ by $\varphi|_{\mathrm{Spin}(n)} = \mathrm{id}$ and $g \cdot \tilde{r_1} \mapsto g \cdot \tilde{r_2}$ on the other component. This gives rise to a group homomorphism because for any $g, h \in \mathrm{Spin}(n)$, we have

$$
\begin{aligned}
\varphi(gh) &= gh = \varphi(g)\varphi(h), \\
\varphi(g \cdot h\tilde{r_1}) &= g \cdot h\tilde{r_2} = \varphi(g)\varphi(h\tilde{r_1}), \\
\varphi(g\tilde{r_1} \cdot h) &= \varphi((g\tilde{r_1}h\tilde{r_1}^{-1})\tilde{r_1}) = (g\tilde{r_1}h\tilde{r_1}^{-1})\tilde{r_2}, \\
&\overset{(1)}{=} g\tilde{r_2}h\tilde{r_2}^{-1} \cdot \tilde{r_2} = \varphi(g\tilde{r_1})\varphi(h), \\
\varphi(g\tilde{r_1}h\tilde{r_1}) &= g\tilde{r_1}h\tilde{r_1} \overset{(2)}{=} g\tilde{r_2}h\tilde{r_2} = \varphi(g\tilde{r_1})\varphi(h\tilde{r_1}).
\end{aligned}
$$

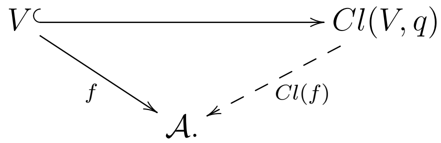
Equation (1) holds because by the second condition, the maps $h \mapsto \tilde{r_1}h\tilde{r_1}$ and $h \mapsto \tilde{r_2}h\tilde{r_2}$ are lifts of c_r with $\tilde{r_i}^2 = -1$ in the Pin^- case and $\tilde{r_i}^2 = 1$ in the Pin^+ case. So they must be equal. The same argument shows $c_{\tilde{r_1}} = c_{\tilde{r_2}}$, so equation (2) also holds.

Since φ is bijective, it is the isomorphism we are looking for. □

The previous definitions for Spin and $\mathrm{Pin}^\pm$ are rather abstract and not very useful in concrete calculations. A concrete realisation is given in terms of Clifford algebras. We only give the definitions and state the theorems relevant for this thesis. Proofs can be found in the excellent literature about this topic, like [ABS64] or [Mei13].

Definition 1.1.5 (Clifford algebra)**.** Let (V, q) be a quadratic vector space, meaning q is a quadratic form on V. The associated Clifford algebra $Cl(V, q)$ is the associative, unital algebra uniquely determined by the following universal property:

Any linear map $f\colon V \to \mathcal{A}$ into an associative algebra $\mathcal{A}$ with unit, that additionally satisfies $f(v)^2 = q(v) \cdot 1_{\mathcal{A}}$, extends uniquely to an algebra homomorphism $Cl(f)\colon Cl(V,q) \to \mathcal{A}$. In terms of diagrams, this means that the diagram given by the solid arrows can be commutatively extended by the dotted arrow



Further, we denote $Cl_{n,0} := Cl(\mathbb{R}^n, -||\cdot||^2)$ and $Cl_{0,n} := Cl(\mathbb{R}^n, ||\cdot||^2)$.

Of course, this definition makes sense over any field k with $\mathrm{char}(k) \neq 2$, but in this thesis we restrict ourselves to the special case $k = \mathbb{R}$.

The universal property of the Clifford algebra has several consequences. For example, any linear map $f\colon (V_1, q_1) \to (V_2, q_2)$ respecting the quadratic forms, i.e. $f^*(q_2) = q_1$, extends uniquely to an algebra homomorphism $Cl(f)\colon Cl(V_1, q_1) \to Cl(V_2, q_2)$. Furthermore, this assignment is functorial.

Definition 1.1.6 (Transposition)**.** Define on $Cl(V,q)$ the new product $x \star y := y \cdot x$. The inclusion $\iota\colon V \hookrightarrow Cl(V,q)$ still satisfy $\iota(v)^2 = q(v)\cdot 1$, so there is a unique algebra homomorphism $(\cdot)^t\colon (Cl(V,q),\cdot) \to (Cl(V,q),\star)$, which also can be seen as a linear map from the Clifford algebra $(CL(V,q),\cdot)$ to itself.

Definition 1.1.7 (Parity operator, even and odd elements)**.** The parity operator Π is defined to be $Cl(-\mathrm{id})$. By functoriality, $\Pi^2 = \mathrm{id}$, so it is an involution. We denote the denote the eigenspace to the eigenvalue 1 and -1, with $Cl^0(V,q)$ and $Cl^1(V,q)$, respectively. An element $x \in Cl(V,q)$ is called *even* if it lies in $Cl^0(V,q)$, and *odd* if it lies in $Cl^1(V,q)$. It is called of pure degree if it is even or odd. To an element x of pure degree we can assign its *degree*, denoted by $|x|$, which is zero, if x is even, and one, if x is odd. The Clifford algebra can be (additively) decomposed into an even and odd part. The set of even elements form a subalgebra.

Definition 1.1.8 (Norm)**.** The norm $N\colon Cl(V,q) \to Cl(V,q)$ is defined by $N(x) := \Pi(x^t)x = (\Pi(x))^t x$. It can be shown that this map actually takes values in the real numbers and is multiplicative [ABS64].

Definition 1.1.9. Let $\varrho\colon Cl(V,q)^\times \to \mathrm{Aut}(Cl(V,q))$ be given by $\varrho(x)(y) = \Pi(x)yx^{-1}$. Define $\Gamma := \{x \in Cl(V,q) \mid \varrho(V) = V\}$. This is a subgroup of $Cl(V,q)^\times$.

Having introduced the previous notation, we are finally able to describe the $\mathrm{Pin}^\pm$ and Spin groups in terms of the Clifford algebras $Cl_{n,0}$ and $Cl_{0,n}$.

Theorem 1.1.10 ([ABS64])**.** *We have the following isomorphisms of groups.*

1. $\mathrm{Pin}^+(n) \cong \{x \in \Gamma | N(x) = 1\} \subseteq Cl_{0,n}$
2. $\mathrm{Pin}^-(n) \cong \{x \in \Gamma | N(x) = 1\} \subseteq Cl_{n,0}$
3. $\mathrm{Spin}(n) \cong \{x \in \Gamma | N(x) = 1, \Pi(x) = x\}$

Furthermore, the groups on the right-hand-side are multiplicatively generated by $\mathbb{R}^n$, considered as a subset of the appropriate Clifford algebra. Therefore, *an element is in* Spin *if and only if it is in* Pin *and a product of an even number of vectors.*

The following observation relates $Cl_{n,0}$ and $Cl_{0,n}$. As mentioned above, a Clifford algebra, considered as a vector space, is generated by elements of pure degree. We define on $Cl_{n,0}$ a new product given by $x * y := (-1)^{|x||y|} x \cdot y$. It is easily verified that this product is associative. Now $\iota\colon \mathbb{R}^n \hookrightarrow (Cl_{n,0})$ is a linear map satisfying $\iota(v)^2 = (-1)v^2 = ||v||^2$ yielding a unique algebra homomorphism $\gamma_+\colon Cl_{0,n} \to Cl_{n,0}$. The same construction works on $Cl_{n,0}$, so we get a unique algebra homomorphism $\gamma_-\colon Cl_{n,0} \to Cl_{0,n}$. We conclude $\gamma_+ \circ \gamma_- = \mathrm{id}_{Cl_{n,0}}$ because it is an algebra homomorphism extending $\iota\colon \mathbb{R}^n \hookrightarrow Cl_{0,n}$. Analogously, we get $\gamma_- \circ \gamma_+ = \mathrm{id}_{Cl_{0,n}}$. Consequently, we have proven the following lemma.

Lemma 1.1.11. *$Cl_{n,0} \cong (Cl_{0,n}, *)$ and $Cl_{0,n} \cong (Cl_{n,0}, *)$, viewed as algebras.*

Let us close this section with the following remarks. The construction for the Spin and $\mathrm{Pin}^\pm$ groups we discussed here was purely topological; the only information we used for the construction were number of path connected components or informations encoded in the fundamental group of the base space. In fact, Theorem 1.1.1 can also be applied to $\mathrm{GL}(n)^+$, which has the same homotopy type as $\mathrm{SO}(n)$ (the polar decomposition yields a strong deformation retract $\mathrm{GL}(n) \to \mathrm{SO}(n)$). Therefore, Theorem 1.1.4 can be generalised to $\mathrm{GL}(n)$. We denote the corresponding groups with $\mathrm{GSpin}(n)$ and $\mathrm{GPin}^\pm(n)$, respectively. These groups actually play a minor role in this thesis. However, they are the underlying reason why the existence of a $\mathrm{Pin}^\pm$ is independent of the choice of a Riemannian metric. This will be explained in the next section.

1.2. Pin structures on bundles

Having introduced the Pin groups, we are in the position to study under which condition a vector bundle possesses a Pin structure. Our base spaces are assumed to be at least paracompact, but the developed theory here will be applied to CW-complexes or compact manifolds only. A great deal of this section is inspired by section 1 of the very worth reading paper [KT90b].

Definition 1.2.1 (Pin structures)**.** Let $O \to B$ be a principal $\mathrm{O}(n)$-bundle. A *$\mathrm{Pin}^\pm$ structure* on O is a reduction to a principal $\mathrm{Pin}^\pm$-bundle. That means that there is a principal $\mathrm{Pin}^\pm$-bundle $P \to B$ and a λ-equivariant map $\rho\colon P \to O$, i.e. $\rho(p \cdot g) = \rho(p) \cdot \lambda(g)$, making the following diagram

$$\begin{array}{ccc} P & \xrightarrow{\rho} & O \\ \downarrow & & \downarrow \\ B & \xrightarrow{\mathrm{id}} & B \end{array}$$

commutative.

A vector bundle $E \to B$ possesses a $\mathrm{Pin}^\pm$ structure if there is a Riemannian metric g on E such that the corresponding orthogonal frame bundle

$O(E) := \{(\mathbb{R}^n, \langle\cdot,\cdot\rangle) \xrightarrow{p} (E_x, g_x) \,|\, p \text{ isometry}\} \to B$ possesses a $\mathrm{Pin}^{\pm}$ structure.

A smooth manifold M possesses a $\mathrm{Pin}^{\pm}$ structure if its tangent bundle possesses a $\mathrm{Pin}^{\pm}$ structure.

Definition 1.2.2 (Equivalence of Pin structures)**.** We call two $\mathrm{Pin}^{\pm}$ structures (P_i, ρ_i) on O *equivalent* if there is an equivariant map $\theta\colon P_1 \to P_2$ making the following diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{\theta} & P_2 \\ \downarrow & & \downarrow \\ O & \xrightarrow{\mathrm{id}} & O \end{array}$$

commutative.

More generally, a map $\Phi\colon O_1 \to O_2$ between two principal $\mathrm{O}(n)$-bundles is called *$Pin^{\pm}$-structure-preserving*, if $\Phi^*\rho_2\colon \Phi^*P_2 \to O_1$ and $\rho_1\colon P_1 \to O_1$ are equivalent structures.

Let us emphasise that the equivariant map ρ is part of the structure and that, although every structure equivalence is a map of principal $\mathrm{Pin}^{\pm}$-bundles, the converse does not hold in general. In particular, different equivariant maps $\rho_i\colon P \to O$ may yield different $\mathrm{Pin}^{\pm}$ structures. An example is given below, see Example 1.2.11.

Since any principal bundle can be represented in terms of 1-cocycles (see Theorem A.1.4), it is desirable, not only from a philosophical perspective but also from a practical one, to have an equivalent descriptions in terms of transition functions. A criterion is given in the next two lemmas.

Lemma 1.2.3. *A principal* $\mathrm{O}(n)$*-bundle* $O \to B$ *has a* $\mathrm{Pin}^{\pm}$ *structure if and only if there is a* $\mathrm{Pin}^{\pm}$*-bundle* $P \to B$*, an open covering* $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ *together with trivialisation maps* $(\Psi_\alpha)_{\alpha\in A}$ *for* P *and* $(\Phi_\alpha)_{\alpha\in A}$ *for* O*, such that the corresponding transition functions are related by*

$$g^O_{\alpha\beta} = \lambda(g^P_{\alpha\beta}),$$

where $\lambda\colon \mathrm{Pin}^{\pm}(n) \to \mathrm{O}(n)$ *is the covering map.*

Proof. "$\Leftarrow$" We define the equivariant map ρ locally via

$$\begin{array}{ccc} P_{U_\alpha} & \xrightarrow{\rho|_{U_\alpha}} & O_{U_\alpha} \\ {\scriptstyle\Psi_\alpha}\downarrow & & \downarrow{\scriptstyle\Phi_\alpha} \\ U_\alpha \times \mathrm{Pin}^{\pm}(n) & \xrightarrow[\mathrm{id}\times\lambda]{} & U_\alpha \times \mathrm{O}(n). \end{array}$$

This map is well defined because, for every $p \in P_{U_\alpha} \cap P_{U_\beta}$, we have

$$\begin{aligned} \Phi_\beta \circ \rho|_{P_{U_\alpha}}(p) &= \Phi_\beta \circ \Phi_\alpha^{-1} \circ (\mathrm{id} \times \lambda) \circ \Psi_\alpha(p) \\ &= (\mathrm{id} \times g^O_{\beta\alpha}) \circ (\mathrm{id} \times \lambda) \circ \Psi_\alpha \circ \Psi_\beta^{-1} \circ \Psi_\beta(p) \\ &= (\mathrm{id} \times g^O_{\beta\alpha}) \circ (\mathrm{id} \times \lambda) \circ (\mathrm{id} \times g^Q_{\alpha\beta})(\pi(p), \bar{p}_\beta) \\ &= (\mathrm{id} \times \lambda) \circ \Psi_\beta \\ &= \Phi_\beta \circ \rho|_{P_{U_\beta}}(p). \end{aligned}$$

Note that ρ is equivariant because all local restrictions are so.

"$\Rightarrow$" Let $P \xrightarrow{\rho} B$ be a $\mathrm{Pin}^{\pm}$ structure of $O \to B$ and $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ an open covering of B, over which P and O trivialise simultaneously. The local trivialisations for P are denoted by Ψ_α, the ones for O by Φ_α. Due to equivariance we have $\Phi_\alpha \circ \rho|_{P_{U_\alpha}} \circ \Psi_\alpha^{-1}(u, g) = (u, \theta_\alpha(u) \cdot \lambda(g))$, where $\theta_\alpha \colon U_\alpha \to \mathrm{O}(n)$ is the map given by $\Phi_\alpha \circ \rho|_{P_{U_\alpha}} \circ \Psi_\alpha^{-1}(u, 1)$. Define new trivialisations for O_{U_α} by $\Phi'_\alpha := (\mathrm{id} \times \lambda) \circ \Phi_\alpha$. It is easy to see that they are indeed equivariant homeomorphisms $O_{U_\alpha} \to U_\alpha \times \mathrm{O}(n)$. Little substitutions show $\Phi'_\alpha \circ \rho|_{U_\alpha} \circ \Psi_\alpha^{-1}(u, q) = (u, \lambda(g))$. □

Note that in the previous lemma we adapted the trivialisations of the orthogonal bundle O to the given bundle P. In practice, we would like to adapt the trivialisations of the $\mathrm{Pin}^{\pm}$ structure to a given set of trivialisations of the given $\mathrm{O}(n)$-bundle. In other words, to a given $\mathrm{O}(n)$-cocycle we want to find a nice $\mathrm{Pin}^{\pm}$-cocycle covering it. The next lemma gives a positive result in the case the base space can be covered by 'good' open subsets, namely path connected and simply-connected. When the base space is a CW-complex or a manifold, we always can find such a covering because they are locally contractible.

Lemma 1.2.4. *Let $P \xrightarrow{\rho} O$ be a $\mathrm{Pin}^{\pm}$ structure and $(U_\alpha)_{\alpha \in A}$ a covering of 'good' open subsets over which P and O trivialises. Let a set of trivialisation maps $\Phi_\alpha \colon P_{U_\alpha} \to U_\alpha \times \mathrm{O}(n)$ be given. Then we can find a set of trivialisation maps $\Psi_\alpha \colon P_{U_\alpha} \to U_\alpha \times \mathrm{Pin}^{\pm}(n)$ making the following diagram*

$$\begin{array}{ccc} P_{U_\alpha} & \xrightarrow{\rho|_{U_\alpha}} & O_{U_\alpha} \\ \downarrow{\scriptstyle \Psi_\alpha} & & \downarrow{\scriptstyle \Phi_\alpha} \\ U_\alpha \times \mathrm{Pin}^{\pm}(n) & \xrightarrow[\mathrm{id} \times \lambda]{} & U_\alpha \times \mathrm{O}(n) \end{array}$$

commutative.

Proof. Let $\tilde{\Psi}_\alpha \colon P_{U_\alpha} \to U_\alpha \times \mathrm{Pin}^{\pm}(n)$ be a set of trivialisation maps. Using the same notation as in the proof of the previous lemma, we have $\Phi_\alpha \circ \rho|_{P_{U_\alpha}} = (\mathrm{id} \times \theta_\alpha(\cdot) \cdot \lambda(\cdot))$. Since U_α is path connected and simply-connected, the map $\theta \colon U_\alpha \to \mathrm{O}(n)$ has a lift $\tilde{\theta}_\alpha \to \mathrm{Pin}^{\pm}(n)$. Now,

$$\Psi_\alpha := \left(\mathrm{id} \times \left(\tilde{\theta}_\alpha(\cdot) \right)^{-1} \right) \circ \tilde{\Psi}_\alpha$$

defines an equivariant homeomorphism. It is straightforward to verify, that those Ψ_α make the diagram in the assertion commutative. □

Lemma 1.2.5. *Let $P_i \xrightarrow{\rho_i} O$ be two $\mathrm{Pin}^{\pm}$ structures on O, $\{o_{\alpha\beta}\}$ a cocycle representing O, and $\{p^i_{\alpha\beta}\}$ cocycles representing P_i and satisfying the relation $\lambda(p^i_{\alpha\beta}) = o_{\alpha\beta}$.*

Then P_1 and P_2 are equivalent Pin^- structures if and only if there is a 0-cocycle $\{x_\alpha\}$ with values in $\mathbb{Z}_2 = \ker \lambda$ such that

$$p^1_{\alpha\beta} = x_\alpha^{-1} p^2_{\alpha\beta} x_\beta.$$

Proof. "$\Rightarrow$" Let $\theta\colon P_1 \to P_2$ be the structure equivalence, (U_α, Φ_α) the trivialisations of O yielding the 1-cocycle $\{o_{\alpha\beta}\}$, and $(U_\alpha, \Psi^i_\alpha)$ the trivialisations of P_i giving $\{p^i_{\alpha\beta}\}$. Further, define the family of $\mathrm{Pin}^\pm(n)$ valued function x_α via

$$\mathrm{id} \times x_\alpha = \Psi^2_\alpha \circ \theta \circ (\Psi^1_\alpha)^{-1}.$$

These functions actually take values in $\ker\lambda$ because of

$$\begin{aligned} \mathrm{id} \times \lambda(x_\alpha) &= \rho_2 \circ \Phi_\alpha \circ \theta \circ (\Psi^1_\alpha)^{-1} \\ &= \Phi_\alpha \circ \mathrm{id} \circ \rho_1 \circ (\Psi^1_\alpha)^{-1} \\ &= \Phi_\alpha \circ (\Phi_\alpha)^{-1} = \mathrm{id} \times \mathrm{id}. \end{aligned}$$

Hence, on $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we have the identity

$$\begin{aligned} \mathrm{id} \times p^1_{\alpha\beta} &= \Phi_\alpha \circ (\Phi_\beta)^{-1} = \Psi^1_\alpha \circ \theta^{-1} \circ \theta \circ (\Psi^1_\beta)^{-1} \\ &= \left(\Psi^1_\alpha \circ \theta^{-1} \circ (\Psi^2_\alpha)^{-1}\right) \circ \left(\Psi^2_\alpha \circ (\Psi^2_\beta)^{-1}\right) \circ \left(\Psi^2_\beta \circ \theta \circ (\Psi^1_\beta)^{-1}\right) \\ &= \left(\mathrm{id} \times x_\alpha^{-1}\right) \circ \left(\mathrm{id} \times p^2_{\alpha\beta}\right) \circ (\mathrm{id} \times x_\beta) \\ &= \mathrm{id} \times \left(x_\alpha^{-1} \circ p^2_{\alpha\beta} \circ x_\beta\right). \end{aligned}$$

"$\Leftarrow$" Define $\theta\colon P_1 \to P_2$ locally defined by

$$\theta|_{U_\alpha} = \left(\Psi^2_\alpha\right)^{-1} \circ \mathrm{id} \times x_\alpha \cdot \circ \Psi^1_\alpha|_{U_\alpha}.$$

This gives an isomorphism of (abstract) $\mathrm{Pin}^\pm$ bundles. Since x_α lies in the kernel of λ, the map θ covers the identity because it does so locally by construction. □

The previous lemmas provide the necessary tools to work out some elementary examples.

Example 1.2.6 (Pin structures on line bundles). Let $L \to B$ be a real line bundle over a CW-complex or a manifold. Choose a Riemannian metric and consider its associated $\mathrm{O}(1)$-principal bundle of orthonormal frames $\mathrm{O}(L)$.

We claim that $\mathrm{O}(L)$ has a $\mathrm{Pin}^+(1)$ structure. Indeed, let $\{g_{\alpha\beta}\}$ be a set of transition functions representing $\mathrm{O}(L)$. We define a lift $h_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Pin}^+(1)$ given by

$$h_{\alpha\beta}(u) = \begin{cases} e_1, & \text{if } g_{\alpha\beta} = -1 \in \mathrm{O}(1), \\ 1, & \text{if } g_{\alpha\beta} = 1 \in \mathrm{O}(1), \end{cases}$$

where e_1 denotes the standard basis of $\mathbb{R} \subseteq Cl_{0,1}$. Clearly, $\lambda(h_{\alpha\beta}) = g_{\alpha\beta}$, so it remains to show that the family $h_{\alpha\beta}$ satisfy the cocycle condition. Since $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$, it follows that precisely zero or two out of these three transition functions are not the identity. However, since $e_1^2 = 1$ and $\mathrm{Pin}^+(1)$ is commutative, both cases yields $h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha} = 1$. Therefore, the cocycle $\{h_{\alpha\beta}\}$ gives a $\mathrm{Pin}^+(1)$ structure on $\mathrm{O}(L)$ due to Lemma 1.2.3.

Example 1.2.7. For any line bundle $L \to B$, the three dimensional vector bundle $3L := L \oplus L \oplus L$ has a $\mathrm{Pin}^-(3)$ structure. Indeed, if $\mathrm{O}(L)$ is represented by the transition functions $\{g_{\alpha\beta}\}$, then $\mathrm{O}(3L)$ is represented by $G_{\alpha\beta} = g_{\alpha\beta} \cdot \mathrm{id}_{\mathbb{R}^3}$. Define $h_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Pin}^-(3)$ via

$$h_{\alpha\beta}(u) = \begin{cases} 1, & \text{if } g_{\alpha\beta}(u) = 1 \in \mathrm{O}(1), \\ e_1e_2e_3, & \text{if } g_{\alpha\beta} = -1 \in \mathrm{O}(1), \end{cases}$$

where $\{e_i\}$ denotes the standard basis of $\mathbb{R}^3 \subseteq Cl_{3,0}$. Since $e_1e_2e_3 \cdot e_k \cdot e_3e_2e_1 = -e_k$ for every $k \in \{1,2,3\}$, we conclude that $h_{\alpha\beta}$ is a lift of $G_{\alpha\beta}$. From $(e_1e_2e_3)^2 = 1$ we conclude as in the previous example that $\{h_{\alpha\beta}\}$ satisfy the cocycle condition. Thus, $3L$ possesses a $\mathrm{Pin}^-(3)$ structure by Lemma 1.2.3.

Example 1.2.8 (oriented vector bundles). An oriented vector bundle $E \to B$ of dimension n possesses a $\mathrm{Pin}^\pm$ structure if and only if it possesses a $\mathrm{Spin}(n)$ structure. Indeed, choose a set of local trivialisations over a good open cover $(U_\alpha, \Phi_\alpha)_{\alpha\in A}$ (we assume that B is either a CW-complex or a manifold), such that the corresponding trivialisation functions take values in $\mathrm{SO}(n)$. By Lemma 1.2.4 we can find local trivialisations for the given $\mathrm{Pin}^\pm$ structure such that the corresponding transition functions $h_{\alpha\beta}$ are related through $g_{\alpha\beta} = \lambda(h_{\alpha\beta})$. But this implies that $h_{\alpha\beta}$ takes values in $\mathrm{Spin}(n)$ and therefore we have a reduction to a $\mathrm{Spin}(n)$-principal bundle. This is precisely the definition of having a $\mathrm{Spin}(n)$ structure.

On the other hand, let $\rho\colon P_{\mathrm{Spin}(n)} \to \mathrm{O}(n)$ be a $\mathrm{Spin}(n)$ structure. Then

$$\rho \times \mathrm{id}\colon P_{\mathrm{Spin}(n)} \times_{\mathrm{Spin}(n)} \mathrm{Pin}^\pm(n) \to \mathrm{O}(E)$$

is a $\mathrm{Pin}^\pm$ structure on E.

Example 1.2.9 (Spin structures on the circle). Let $\mathrm{SO}(S^1) = S^1 \times \{e\}$ be the bundle of oriented orthonormal frames. There are two different $\mathrm{Spin}(1)$ structures on $\mathrm{SO}(S^1)$. The first one is the trivial $\mathrm{Spin}(1)$ structure

$$\mathrm{id} \times \lambda\colon S^1 \times \mathrm{Spin}(1) \to S^1 \times \mathrm{SO}(1)$$

and the second one is given by the connected two sheeted covering of S^1,

$$\begin{aligned} (\cdot)^2\colon S^1 &\to S^1, \\ z &\mapsto z^2. \end{aligned}$$

These structures are obviously non-equivalent, because the second one is while the trivial one is not. The circle endowed with the non trivial $\mathrm{Spin}(1)$ structure is denoted by S^1_{Lie}.

Now we want to determine the obstruction for existence of a $\mathrm{Pin}^\pm$ structure. To this end we use a generalisation of Čech-cohomology to non-abelian groups. Its definition and properties can be found in the appendix, see Theorem A.1.4.

Theorem 1.2.10 (Obstruction for Pin structures)**.** *Let $E \to B$ be a vector bundle over a paracompact base space and w_i be the i-th Stiefel-Whitney class. Then E possesses*

- *a Pin^+ structure if and only if $w_2(E) = 0$*
- *a Pin^- structure if and only if $(w_2 - w_1^2)(E) = 0$.*

In particular, the property of possessing a $\mathrm{Pin}^{\pm}$ *structure does not depend on the choice of the Riemannian metric.*

If E possesses a $\mathrm{Pin}^{\pm}$ *structure and if B is a CW-complex or a manifold, then $H^1(X,\mathbb{Z}_2)=\check{H}(X,\mathbb{Z}_2)$ acts simply and transitively on the set of equivalence classes of* $\mathrm{Pin}^{\pm}$ *structures.*

Proof. The short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}^{\pm}(n) \xrightarrow{\lambda} \mathrm{O}(n) \longrightarrow 1$$

gives rises to a long exact sequence

$$\ldots \longrightarrow H^1(B,\mathbb{Z}_2) \longrightarrow H^1(B,\mathrm{Pin}^{\pm}) \xrightarrow{\lambda_*} H^1(B,\mathrm{O}(n)) \xrightarrow{\delta^1} H^2(B,\mathbb{Z}_2)$$

since $\mathbb{Z}_2$ is central in $\mathrm{Pin}^{\pm}(n)$. Any $\mathrm{O}(n)$-valued 1-cocycle corresponds to a principal $\mathrm{O}(n)$-bundle. Thus, for every $x \in H^1(B,\mathrm{O}(n))$, there exists a classifying map $f\colon B \to B\mathrm{O}(n)$ such that $f^*E\mathrm{O}(n)=x$ and we have the following commutative diagram

$$\begin{array}{ccccccc}
\ldots \longrightarrow & H^1(B,\mathrm{Pin}^{\pm}(n)) & \xrightarrow{\lambda_*} & H^1(B,\mathrm{O}(n)) & \xrightarrow{\delta^1_{\pm}} & H^2(B,\mathbb{Z}_2) \\
& \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\
\ldots \longrightarrow & H^1(B\mathrm{O}(n),\mathrm{Pin}^{\pm}(n)) & \xrightarrow{\lambda_*} & H^1(B\mathrm{O}(n),\mathrm{O}(n)) & \xrightarrow{\delta^1_{\pm}} & H^2(B\mathrm{O}(n),\mathbb{Z}_2)
\end{array}$$

with exact rows. It is well known that $H^2(B\mathrm{O}(n),\mathbb{Z}_2)\cong\mathbb{Z}_2[w_1,\ldots,w_n]$, considered as graded rings. Therefore $\delta^1_{\pm}(E\mathrm{O}(n))=\mu_1^{\pm}w_1^2+\mu_2^{\pm}w_2$ is the universal obstruction for a $\mathrm{Pin}^{\pm}$ structure. We determine the coefficients by considering special examples.

Firstly, observe that $\mu_2^{\pm}$ must not be zero, otherwise every oriented vector bundle would possess a Spin structure, which is known to be false. For example, the universal bundle over $B\mathrm{SO}(2)$ cannot carry a $\mathrm{Spin}(2)$ structure. Next we conclude from Example 1.2.6 that $\mu_1^{+}=0$, because there exists non-orientable line bundles, like the tautological line bundle over $\mathbb{R}P^2$. Recall that for every line bundle L, the vector bundle $3L$ possesses a Pin^{-} structure. Since $w_2(3L)=3w_1(L)^2=w_1(L)^2$ and $w_1(3L)^2=3^2w_1(L)^2=w_1(L)^2$, we conclude $(w_2-w_1^2)(3L)=0$. But, for the tautological line bundle $\gamma_2\to\mathbb{R}P^2$, we know $w_1(\gamma_2)\neq 0$, and derive therefore $\mu_1^{-}\neq 0$.

Summarising: We have deduced that $\delta^1_{-}=w_2+w_1^2$ and that $\delta^1_{+}=w_2$.

A very nice way to see that the existence of a $\mathrm{Pin}^{\pm}$ structure is independent of the Riemannian metric is to use the generalised Pin group from the previous section. From the commutative diagram

$$\begin{array}{ccccc}
H^1(B,\mathrm{Pin}^{\pm}(n)) & \xrightarrow{\lambda_*} & H^1(B,\mathrm{O}(n)) & \xrightarrow{\delta^1_{\pm}} & H^2(B,\mathbb{Z}_2) \\
\downarrow{\scriptstyle \mathrm{incl}_*} & & \downarrow{\scriptstyle \mathrm{incl}_*} & & {\scriptstyle\cong}\downarrow{\scriptstyle \mathrm{id}} \\
H^1(B,\mathrm{GPin}^{\pm}) & \xrightarrow{\lambda_*} & H^1(B,\mathrm{GL}(n)) & \xrightarrow{\Delta^1_{\pm}} & H^2(B,\mathbb{Z}_2)
\end{array}$$

with exact rows we conclude that any reduction $\mathrm{O}(E)\subseteq\mathrm{GL}(E)$ carries a $\mathrm{Pin}^{\pm}$ structure if and only if $\mathrm{GL}(E)$ carries a $\mathrm{GPin}^{\pm}$ structure. But the latter is independent of a Riemannian metric.

Now let $\mathrm{O}(E) \to B$ be represented by the cocycle $\{g_{\alpha\beta}\}$. The action of $H^1(B, \mathbb{Z}_2) = \check{H}^1(B, \mathbb{Z}_2)$ on the equivalence classes of $\mathrm{Pin}^\pm$ structures is given as follows. Let $x \in \check{H}^1(B, \mathbb{Z}_2)$ be represented by the $\mathbb{Z}_2$-valued cocycle $\{x_{\alpha\beta}\}$ and a $\mathrm{Pin}^\pm$ structure P of $\mathrm{O}(E)$ be represented by $\{h_{\alpha\beta}\}$ such that $\lambda(h_{\alpha\beta}) = g_{\alpha\beta}$ (after a restriction to a refinement of 'good' open subsets, we may assume that all three different cocycles live on the same domain). Then the action is given by $x.P := P'$, where P' is the unique $\mathrm{Pin}^\pm$ structure represented by $\{x_{\alpha\beta} \cdot h_{\alpha\beta}\}$. Note that $\{x_{\alpha\beta} \cdot h_{\alpha\beta}\}$ is a cocycle, since $x_{\alpha\beta}$ takes values in the centre of $\mathrm{Pin}^\pm(n)$.

This indeed defines an action because $(x+y).P = \{x_{\alpha\beta}y_{\alpha\beta}h_{\alpha\beta}\} = x.(y.P)$ and $0.P = \{1_{\alpha\beta}h_{\alpha\beta}\} = \{h_{\alpha\beta}\} = P$.

One can see that this action is free by the following argument. Assume that $x = \{x_{\alpha\beta}\} \in \check{H}^1(B, \mathbb{Z}_2)$ maps P represented by $\{h_{\alpha\beta}\}$ to an equivalent structure P'. Lemma 1.2.5 implies that the cocycles $\{x_{\alpha\beta}h_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ are cohomologous by a coboundary with values in $\ker(\lambda)$. In other words, there is a family of continuous $\ker(\lambda)$-valued functions $\{f_\alpha\}$ such that $x_{\alpha\beta}h_{\alpha\beta} = f_\alpha h_{\alpha\beta} f_\beta^{-1}$. Since $\ker(\lambda)$ is central in $\mathrm{Pin}^\pm$, this is equivalent to $\{x_{\alpha\beta}\} = \{f_\alpha f_\beta^{-1}\}$. Thus, x is itself a coboundary, so $x = 0 \in \check{H}^1(B, \mathbb{Z}_2)$.

To verify transitivity, let P and P' be two $\mathrm{Pin}^\pm$ structures on $\mathrm{O}(E)$. Again, we pick trivialisations for $\mathrm{O}(E)$, P, and P' such that the corresponding transition function of these bundles are related by $g_{\alpha\beta} = \lambda(h_{\alpha\beta}) = \lambda(h'_{\alpha\beta})$. Then $\{x_{\alpha\beta}\} := \{h_{\alpha\beta}(h'_{\alpha\beta})^{-1}\}$ is a set of continuous functions $U_\alpha \cap U_\beta \to \ker(\lambda)$. We need to verify that $\{x_{\alpha\beta}\}$ satisfies the cocycle condition. This follows from

$$\begin{aligned} 1 = h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha} &= x_{\alpha\beta}h'_{\alpha\beta}x_{\beta\gamma}h'_{\beta\gamma}x_{\gamma\alpha}h'_{\gamma\alpha} \\ &= x_{\alpha\beta}x_{\beta\gamma}x_{\gamma\alpha}h'_{\alpha\beta}h'_{\beta\gamma}h'_{\gamma\alpha} \\ &= x_{\alpha\beta}x_{\beta\gamma}x_{\gamma\alpha}. \end{aligned}$$

This shows $x.P' = P$ and we are done. □

This theorem has many implications. For instance, there are Spin structures on a given $\mathrm{SO}(n)$ bundle which are not equivalent, but isomorphic as abstract Spin bundles. This observation extends to $\mathrm{Pin}^\pm$ bundles.

Example 1.2.11. Since $H^1(S^1, \mathbb{Z}_2) \cong \mathbb{Z}_2$, there are two non-equivalent Spin(2) structure on the trivial bundle $S^1 \times \mathrm{SO}(2) \xrightarrow{\mathrm{pr}_1} S^1$. However, since Spin(2) is connected, we have $\pi_1(B\mathrm{Spin}(2)) \cong \pi_0(\mathrm{Spin}(2)) = 0$. By the universal property of $B\mathrm{Spin}(2)$, every Spin(2)-principal bundle on S^1 is necessarily trivial.

Corollary 1.2.12. *A vector bundle $E \to B$ carries a* Spin *structure if and only if $w_2(E) = w_1(E) = 0$.*

Proof. It is known that a vector bundle is orientable if and only if its first Stiefel-Whitney class $w_1(E)$ vanishes. The rest is a reformulation of the fact that an orientable vector bundle has a Spin structure if and only if it has a $\mathrm{Pin}^\pm$ structure, see Example 1.2.8. □

Corollary 1.2.13. *Let E and F be two vector bundles over B.*

1. *If E carries a* $\mathrm{Pin}^\pm$ *structure and F a* Spin *structure, then $E \oplus F$ carries a* $\mathrm{Pin}^\pm$ *structure.*
2. *If E carries a* $\mathrm{Pin}^\pm$ *structure and $E \oplus F$ is trivial, then F carries a* $\mathrm{Pin}^\mp$ *structure.*

Proof. To prove the first assertion, use Cartan's formula and that F is Spin to derive

$$w_2(E \oplus F) = w_2(E) + w_1(E)w_1(F) + w_2(F) = w_2(E)$$

and

$$w_1(E \oplus F) = w_1(E) + w_1(F) = w_1(E).$$

Therefore, E and $E \oplus F$ have the same cohomology class as obstruction.

The proof of the second point is similar; just use

$$0 = w_1(E \oplus F) = w_1(E) + w_1(F)$$

and Cartan's formula to derive

$$0 = w_2(E \oplus F) = w_2(E) + w_1(F)^2 + w_2(F).$$

Now, if E carries a Pin^+ structure, then F carries a Pin^- structure, and vice versa. □

The two-out-of-three lemma states that if two of the three vector bundles E, F, and $E \oplus F$ over the same base space carry a Spin structure, so does the third. The choice of the two Spin structures uniquely determines the third, see [Mil63]. This lemma does not generalise to $\mathrm{Pin}^\pm$ structures in general. For example, $\mathbb{R}P^2 \times \mathbb{R}P^2$ does not carry a Pin^- structure although $\mathbb{R}P^2$ does. However, if the vector bundles have the same first Stiefel-Whitney class, we have a generalisation to $\mathrm{Pin}^\pm$ structures.

Lemma 1.2.14 (two-out-of-three)**.** *Let $E_1, E_2 \to B$ be two vector bundles with $w_1(E_1) = w_1(E_2)$. Then the following three statements hold.*

1. *A* Pin^- *structure on E_1 and a* Spin *structure on $E_1 \oplus E_2$ uniquely determine a* Pin^+ *structure on E_2.*
2. *A* Pin^+ *structure on E_2 and a* Spin *structure on $E_1 \oplus E_2$ uniquely determine a* Pin^- *structure on E_1.*
3. *A* Pin^- *structure on E_1 and a* Pin^+ *structure on E_2 uniquely determines a* Spin *structure on $E_1 \oplus E_2$.*

Proof. Since the three proofs only differ by minimal modifications, we only prove the first statement. Let E_1 and E_2 be trivialised over the open cover $(U_\alpha)_{\alpha \in A}$ by Φ^1_α and Φ^2_α yielding transition functions $\{g^{(1)}_{\alpha\beta}\}$ and $\{g^{(2)}_{\alpha\beta}\}$, respectively. Then $E_1 \oplus E_2$ is also trivial over this open cover and has the transition functions $\{g^{(1)}_{\alpha\beta} \oplus g^{(2)}_{\alpha\beta}\}$. Since $w_1(E \oplus F) = w_1(E_1) + w_1(E_2) = 2w_1(E_1) = 0$ the bundle $E \oplus F$ is orientable and we can arrange that $\{g^{(1)}_{\alpha\beta} \oplus g^{(2)}_{\alpha\beta}\} \in \mathrm{SO}(n_1 + n_2)$ by changing some trivialisation maps of E_1 if necessary. Indeed, if the frame $\{\Phi^1_\alpha \oplus \Phi^2_\alpha(\cdot, e_j)\}_{1 \le j \le n_1+n_2}$ has not the chosen orientation, then we replace Φ^1_α by $\mathrm{id} \times \mathrm{diag}(-1, 1, \dots, 1) \circ \Phi^1_\alpha$.

Let $\{h_{\alpha\beta}\}$ be a fixed Spin structure with $\lambda(h_{\alpha\beta}) = \{g^{(1)}_{\alpha\beta} \oplus g^{(2)}_{\alpha\beta}\}$. We can decompose $h_{\alpha\beta} = u_{\alpha\beta} \cdot v_{\alpha\beta}$, where $\{u_{\alpha\beta}\}$ is a cocycle representing the chosen Pin^- structure of E_1 lifting the cocycle $\{g_{\alpha\beta}\}$ and $\{v_{\alpha\beta}\}$ is a family of continuous functions with values in Cl_{0,n_2} lifting the transition functions $\{g^{(2)}_{\alpha\beta}\}$.

We have to show that $\{v_{\alpha\beta}\}$ satisfies the cocycle condition in Cl_{0,n_2}. To this end, we use the alternated product on $Cl_{n_2,0}$ defined in Corollary 1.1.11 to derive the statement from the cocycle condition of $u_{\alpha\beta}$ and $h_{\alpha\beta}$ via the following calculation:

$$\begin{aligned} 1 = h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha} &= u_{\alpha\beta}v_{\alpha\beta}u_{\beta\gamma}v_{\beta\gamma}u_{\gamma\alpha}v_{\gamma\alpha} \\ &= \underbrace{u_{\alpha\beta}u_{\beta\gamma}u_{\gamma\alpha}}_{=1} v_{\alpha\beta}v_{\beta\gamma}v_{\gamma\alpha}(-1)^{|u_{\alpha\gamma}|\cdot|v_{\alpha\beta}\cdot v_{\beta\gamma}|}(-1)^{|u_{\beta\gamma}||v_{\alpha\beta}|} \\ &\overset{(1)}{=} v_{\alpha\beta}v_{\beta\gamma}v_{\gamma\alpha}(-1)^{|v_{\alpha\gamma}|\cdot|v_{\alpha\beta}\cdot v_{\beta\gamma}|}(-1)^{|v_{\beta\gamma}||v_{\alpha\beta}|} \\ &= v_{\alpha\beta} * v_{\beta\gamma} * v_{\gamma\alpha}. \end{aligned}$$

Note that equality (1) holds because $\det g^{(1)}_{\alpha\beta} \cdot \det g^{(2)}_{\alpha\beta} > 0$ and therefore $u_{\alpha\beta} \in \mathrm{Spin} \Leftrightarrow v_{\alpha\beta} \in \mathrm{Spin}$. But this is equivalent to saying that $u_{\alpha\beta}$ and $v_{\alpha\beta}$ have the same degree. □

We have already seen that the existence of one $\mathrm{Pin}^\pm$ structure implies the existence of $\#H^1(B, \mathbb{Z}_2)$ many different non-equivalent $\mathrm{Pin}^\pm$ structures as long as B is locally contractible. However, the bijection between the set of non-equivalent $\mathrm{Pin}^\pm$ structure over a bundle and $H^1(B; \mathbb{Z}_2)$ is not canonical. Corollary 1.2.13 gives us a bijection between the $\mathrm{Pin}^\pm$ structures on E and the $\mathrm{Pin}^\pm$ structures on $E \oplus \varepsilon^r$. But again, this bijection is not canonical.

The next lemma strengthens this observation by providing a canonical bijection.

Lemma 1.2.15. *For any vector bundle $E \to B$ of rank n and any $r \geq 0$ there is a canonical bijection between the equivalence classes of* $\mathrm{Pin}^\pm$ *structures on E and and the equivalence classes of* $\mathrm{Pin}^\pm$ *structures on $E \oplus \varepsilon^r$. This bijection is natural with respect to isometric vector bundle homomorphisms.*

Proof. Since $\mathrm{O}(E\oplus\varepsilon^r) \cong \mathrm{O}(E)\times_{\mathrm{O}(n)}\mathrm{O}(n+r)$, every $\mathrm{Pin}^\pm$ structure (P, ρ) over $\mathrm{O}(E)$ induces a $\mathrm{Pin}^\pm$ structure on $\mathrm{O}(E\oplus\varepsilon^r)$ via $(P\times_{\mathrm{Pin}^\pm(n)}\mathrm{Pin}^\pm(n+r), \rho\times\lambda)$.

Clearly, this assignment preserves structure equivalences because a structure equivalence

$$\begin{array}{ccc} P & \xrightarrow{\theta} & P' \\ \downarrow & & \downarrow \\ \mathrm{O}(E) & \xrightarrow{\mathrm{id}} & \mathrm{O}(E) \end{array}$$

gives rise to a structure equivalence

$$\begin{array}{ccc} P \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+r) & \xrightarrow{\theta\times\mathrm{id}} & P' \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+r) \\ \downarrow & & \downarrow \\ \mathrm{O}(E) \times_{\mathrm{O}(n)} \mathrm{O}(n+r) & \xrightarrow{\mathrm{id}} & \mathrm{O}(E) \times_{\mathrm{O}(n)} \mathrm{O}(n+r). \end{array}$$

So, this map descends to a well-defined map between the sets of equivalence classes of $\mathrm{Pin}^\pm$ structures on the given bundles.

Let us prove injectivity first. If $\{g_{\alpha\beta}\}$ is a set of transition functions describing $\mathrm{O}(E)$, then $\iota(g_{\alpha\beta})$ describes $\mathrm{O}(E) \times_{\mathrm{O}(n)} \mathrm{O}(n+r)$, where $\iota\colon \mathrm{O}(n) \to \mathrm{O}(n+r)$ is the canonical inclusion. The same is true for P and $P \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+r)$. Now consider two equivalent $\mathrm{Pin}^\pm$ structure $P \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+r)$ and $P' \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+r)$ over $\mathrm{O}(E) \times_{\mathrm{O}(n)} \mathrm{O}(n+r)$. By Lemma 1.2.4 these two $\mathrm{Pin}^\pm$ structures can be represented by cocycles $\{H_{\alpha\beta}\}$ and $\{H'_{\alpha\beta}\}$ such that $\{\iota(g_{\alpha\beta})\} = \{\lambda(H_{\alpha\beta})\} = \{\lambda(H'_{\alpha\beta})\}$. Consequently, the cocycles $\{H_{\alpha\beta}\}$ and $\{H'_{\alpha\beta}\}$ take values in $\iota(\mathrm{Pin}^\pm(n)) \subseteq \mathrm{Pin}^\pm(n+r)$. Therefore, there exist $\mathrm{Pin}^\pm(n)$-valued cocycles $\{h_{\alpha\beta}\}$ and $\{h'_{\alpha\beta}\}$ such that $\{H_{\alpha\beta}\} = \{\iota(H_{\alpha\beta})\}$ and $\{H'_{\alpha\beta}\} = \{\iota(H'_{\alpha\beta})\}$, respectively. Since the chosen $\mathrm{Pin}^\pm$ structures are equivalent, there exists a family of $\ker(\lambda)$-valued continuous functions $\{x_\alpha\}$ such that $x_\alpha H'_{\alpha\beta} x_\beta^{-1} = H_{\alpha\beta}$. By the injectivity of ι, this implies that $\{x_\alpha h'_{\alpha\beta} x_\beta^{-1}\} = h_{\alpha\beta}$. Since, by Lemma 1.2.4, $\{h_{\alpha\beta}\}$ and $\{h'_{\alpha\beta}\}$ are the transition functions of P and P', respectively, we conclude that P must be equivalent to P'. Injectivity is therefore proven.

Next we address surjectivity. Let P be a $\mathrm{Pin}^\pm$ structure for $\mathrm{O}(E \oplus \varepsilon^r)$ and let $\{\iota(g_{\alpha\beta})\}$ be a set of transition functions coming from $\mathrm{O}(E)$. Again, by Lemma 1.2.4 we find trivialisations for P such that the corresponding transition functions $\{H_{\alpha\beta}\}$ satisfy $\{\lambda(H_{\alpha\beta})\} = \{\iota(g_{\alpha\beta})\}$. So, there exists a cocycle $\{h_{\alpha\beta}\}$ with values in $\mathrm{Pin}^\pm(n) \xrightarrow{\iota} \mathrm{Pin}^\pm(n+r)$ such that $h_{\alpha\beta} = H_{\alpha\beta}$. But this is equivalent to the existence of a $\mathrm{Pin}^\pm(n)$ structure Q such that $Q \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+r)$ and P are equivalent $\mathrm{Pin}^\pm(n+r)$ structure over $\mathrm{O}(E) \times_{\mathrm{O}(n)} \mathrm{O}(n+r)$.

It remains to show naturality with respect to isometric bundle maps. First note that an isometry $f\colon E_1 \to E_2$ induces a morphism of principal $\mathrm{O}(n)$-bundle $F = \mathrm{O}(f)\colon \mathrm{O}(E_1) \to \mathrm{O}(E_2)$. Because f is an isometry, so is $g := f \oplus \mathrm{id}_{\varepsilon^r}$ if we equip $E_1 \oplus \varepsilon^r$ and $E_1 \oplus \varepsilon^r$ with the product metric.

Since

$$\begin{array}{ccc} \mathrm{O}(E_1 \oplus \varepsilon^r) & \xleftarrow{\;\cong\;} & \mathrm{O}(E) \times_{\mathrm{O}(n)} \mathrm{O}(n+r) \\ {\scriptstyle \mathrm{O}(g)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{O}(f)\times \mathrm{id}} \\ \mathrm{O}(E \oplus \varepsilon^r) & \xleftarrow{\;\cong\;} & \mathrm{O}(E_2) \times_{\mathrm{O}(n)} \mathrm{O}(n+r) \end{array}$$

commutes, we can work with the right-hand-side instead. The claim follows now from the observation that $(\mathrm{O}(f) \times \mathrm{id})^* \left(P_2 \times_{\mathrm{Pin}^\pm} \mathrm{Pin}^\pm(n+r)\right)$ and $\mathrm{O}(f)^*(P_2) \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+r)$ are equivalent $\mathrm{Pin}^\pm$ structures on $\mathrm{O}(E) \times_{\mathrm{O}(n)} \mathrm{O}(n+r)$. □

Remark 1.2.16. Notice that the restriction to isometric bundle isomorphism is only a restriction if we care about a special Riemannian metric; something we do not do in this thesis. However, it is nice to have a criterion at hand that is independent of the choice of a particular metrics or, equivalently, independent of the choice of some $\mathrm{O}(n)$-reduction.

Let $\rho_j\colon P_j \to B_j$ be two GL_n principal bundles over CW-complexes, $O_j \subseteq P_j$ be two $\mathrm{O}(n)$-reductions, and $Q_j \xrightarrow{\rho_j} O_j$ be two $\mathrm{Pin}^\pm$ structures. We call a map of principal GL_n-bundles *Pin-structure-preserving*, if

$$Q_1 \times_{\mathrm{Pin}^\pm} \mathrm{GPin}^\pm \cong \Phi^* \left(Q_2 \times_{\mathrm{Pin}^\pm} \mathrm{GPin}^\pm\right)$$

as $\mathrm{GPin}^\pm$ structures. By Theorem 1.2.10, these $\mathrm{GPin}^\pm$ structures depend only on the $\mathrm{Pin}^\pm$ structure and not on the chosen $\mathrm{O}(n)$-reduction. Thus, we do not have to restrict ourselves to isometric vector bundle maps.

A nice application of the stabilisation property is that we can descend $\mathrm{Pin}^\pm$ structures from manifolds to submanifolds under certain conditions. Indeed, if $V \subseteq M$ is a manifold with trivial normal bundle, we can use the isomorphism

$$\psi\colon TM|V = TV \oplus \nu(V \hookrightarrow M) \xrightarrow{\cong} TV \oplus \varepsilon^{\mathrm{codim} V}$$

to pull back the $\mathrm{Pin}^\pm$ structure from $TM|V$ to $TV \oplus \varepsilon^r$ via ψ^{-1} and then apply the stabilisation Lemma 1.2.15 to induce a $\mathrm{Pin}^\pm$ structure on TV. Keep in mind that those $\mathrm{Pin}^\pm$ structures might depend on the choice of ψ.

Example 1.2.17. An example of great importance is the case $V = \partial M$ because the boundary always has a trivial normal bundle. By the observation from above we have to make a choice how to trivialise the normal bundle. Our convention will be 'inward normal last', which means the following: After we have chosen a Riemannian metric on M, we take the unique isometry

$$\mathrm{id} \oplus \varphi\colon TM|_{\partial M} \to T\partial M \oplus \varepsilon,$$

which sends the inward pointing normal vector of length 1 to the constant the constant unit section $1\colon M \to \varepsilon$.

Note that a choice of a Riemannian metric gives a decomposition of the tangent bundle at the boundary

$$TM|_{\partial M} = T\partial M \oplus T\partial M^\perp = T\partial M \oplus \nu$$

and therefore determines φ up to a sign. Conversely, a vector bundle map $\mathrm{id} \oplus \varphi$ and a Riemannian metric on $T\partial M$ induce a unique Riemannian metric on $TM|_{\partial M}$, which can be extended not uniquely to M. So, we could also start with a map φ, which sends the unit section to a inward pointing nowhere vanishing vector vectorfield.

This vector bundle isometry induces an equivariant map on the associated bundles of orthonormal frames

$$\mathrm{O}(\varphi)\colon \mathrm{O}(TM_{\partial M}) \to \mathrm{O}(T\partial M) \times_{\mathrm{O}(n)} \mathrm{O}(n+1) \cong \mathrm{O}(T\partial M \oplus \varepsilon).$$

The latter isomorphism is canonical and given by

$$[p, A] \mapsto \left(e_i \mapsto \left(p\left(\sum^n a_{j,i} e_j\right), a_{n+1,i}\right)\right).$$

Now we can pull the the $\mathrm{Pin}^\pm$ structure back from the left-hand-side with $\mathrm{O}(\varphi)^{-1} = \mathrm{O}(\varphi^{-1})$ and apply Lemma 1.2.15.

Example 1.2.18. Let M be an n-dimensional $\mathrm{Pin}^\pm$ manifold and fix a $\mathrm{Pin}^\pm$ structure $P \xrightarrow{\rho} \mathrm{O}(M)$. If we endow $M \times I$ with the product metric, the chosen $\mathrm{Pin}^\pm$ structure on M induces canonically a Pin structure on $M \times I$ as follows. We have an isometry

$$\begin{aligned}\varphi\colon \mathrm{pr}_1^*TM \oplus \varepsilon &\to T(M \times I)\\ (v, \lambda) &\mapsto v + \lambda \cdot \partial_t,\end{aligned}$$

which gives rise to an equivariant map

$$\mathrm{pr}_1^*\mathrm{O}(M) \times_{\mathrm{O}(n)} \mathrm{O}(n+1) = \mathrm{O}(\mathrm{pr}_1^*TM \oplus \varepsilon) \xrightarrow{\cong} \mathrm{O}(M \times I).$$

The $\mathrm{Pin}^\pm$ structure on $M \times I$ is defined as the pullback

$$\begin{array}{ccc}\mathrm{O}(\varphi^{-1})^*(P \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+1)) & \longrightarrow & P \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+1)\\ \downarrow & & \downarrow{\scriptstyle \rho\times\mathrm{id}}\\ \mathrm{O}(M \times I) & \xrightarrow{\mathrm{O}(\varphi^{-1})} & \mathrm{O}(\mathrm{pr}_1^*TM \oplus \varepsilon)\end{array}$$

and will be denoted with $Q \to \mathrm{O}(M \times I)$.

How does the $\mathrm{Pin}^\pm$ descend to the boundary $\partial(M \times I) = M \times \partial I$? The isometry $\varphi_0 := \varphi|_{M\times\{0\}}$ sends the constant section $(0, 1)$ to ∂_t, so it fits the 'inward pointing last' convention. From

$$\begin{array}{ccc}\mathrm{O}(\varphi_0)^*(Q) & \longrightarrow & P \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+1)\\ \downarrow & & \downarrow{\scriptstyle \rho\times\mathrm{id}}\\ \mathrm{O}(TM \times \{0\} \oplus \varepsilon) & \xrightarrow{\mathrm{id}} & \mathrm{O}(\mathrm{pr}_1^*TM \oplus \varepsilon),\end{array}$$

we conclude that the $\mathrm{Pin}^\pm$ structure induced from $M \times I$ on $M \times \{0\}$ is equivalent to the original one.

On $M \times \{1\}$ we have a different situation. Because $\varphi(0, 1)$ is an outward pointing vector field, we have to use the isometry

$$\varphi_1\colon \mathrm{pr}_1{}^*(TM)|_{M\times\{1\}} \oplus \varepsilon \to T(M \times I)|_{M\times\{1\}}.$$

Define

$$\begin{aligned}\psi\colon \mathrm{O}(TM) \times_{\mathrm{O}(n)} \mathrm{O}(n+1) &\to \mathrm{O}(TM) \times_{\mathrm{O}(n)} \mathrm{O}(n+1),\\ [p, A] &\mapsto [p, \mathrm{diag}(1, \dots, 1, -1)A].\end{aligned}$$

Then a model for the pullback

$$\begin{array}{ccc}\mathrm{O}(\varphi_1)^*Q|_{M\times\{1\}} & \longrightarrow & P \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+1)\\ \downarrow & & \downarrow\\ \mathrm{O}(TM \times \{1\} \oplus \varepsilon) & \xrightarrow[\mathrm{O}(\varphi_0^{-1}\circ\varphi_1)]{} & \mathrm{O}(TM \oplus \varepsilon)\\ \downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow\\ \mathrm{O}(TM) \times_{\mathrm{O}(n)} \mathrm{O}(n+1) & \xrightarrow[\psi]{} & \mathrm{O}(TM) \times_{\mathrm{O}(n)} \mathrm{O}(n+1)\end{array}$$

is given by

$$\begin{array}{ccc} P\times_{\mathrm{Pin}^\pm(n)}\mathrm{Pin}^\pm(n+1) & \xrightarrow{\theta} & P\times_{\mathrm{Pin}^\pm(n)}\mathrm{Pin}^\pm(n+1) \\ \downarrow{\scriptstyle \rho\times\mathrm{id}} & & \downarrow{\scriptstyle \rho\times\mathrm{id}} \\ \mathrm{O}(TM)\times_{\mathrm{O}(n)}\mathrm{O}(n+1) & \xrightarrow{\psi} & \mathrm{O}(TM)\times_{\mathrm{O}(n)}\mathrm{O}(n+1), \end{array}$$

where θ is given by $[q,v]\mapsto[q,e_{n+1}v]$.

We know that $H^1(M,\mathbb{Z}_2)$ acts simply and transitively on the $\mathrm{Pin}^\pm$ structures of M. The element transferring

$$P\times_{\mathrm{Pin}^\pm(n)}\mathrm{Pin}^\pm(n+1)\xrightarrow{\rho\times\mathrm{id}}\mathrm{O}(TM)\times_{\mathrm{O}(n)}\mathrm{O}(n+1)$$

to

$$P\times_{\mathrm{Pin}^\pm(n)}\mathrm{Pin}^\pm(n+1)\xrightarrow{\psi\circ\rho\times\mathrm{id}}\mathrm{O}(TM)\times_{\mathrm{O}(n)}\mathrm{O}(n+1)$$

is the first Stiefel-Whitney class.

Indeed, if $\{g_{\alpha\beta}\}$ is a cocycle representing $\mathrm{O}(TM)$ and $\{h_{\alpha\beta}\}$ is a cocycle representing $(P\times_{\mathrm{Pin}^\pm(n)}\mathrm{Pin}^\pm(n+1),\rho\times\mathrm{id})$ with $\lambda(h_{\alpha\beta})=g_{\alpha\beta}$, then it follows from the commutative diagram above that the induced $\mathrm{Pin}^\pm$ structure on $M\times\{1\}$ can be represented by the cocycle $\{e_{n+1}h_{\alpha\beta}e_{n+1}^{-1}\}=\{(-1)^{|h_{\alpha\beta}|}h_{\alpha\beta}\}$. But $\{(-1)^{|h_{\alpha\beta}|}\}=\{\det(g_{\alpha\beta})\}$ is the cocycle representing the first Stiefel-Whitney class.

Recapitulatory, one can say that the induced $\mathrm{Pin}^\pm$ structures on $M\times\{0\}$ and $M\times\{1\}$ differ by the action of the first Stiefel-Whitney class $w_1(M)$.

Definition 1.2.19. If $P\xrightarrow{\rho}\mathrm{O}(TM)$ is a $\mathrm{Pin}^\pm$ structure on M, we call $w_1(M).P$ the *inverse $\mathrm{Pin}^\pm$ structure*. If we have fixed a $\mathrm{Pin}^\pm$ structure on M, we denote with $\bar{M}$ the same manifold but endowed with the inverse $\mathrm{Pin}^\pm$ structure.

Besides from the stabilisation property there is another bijection theorem relating Pin^- structures on a vector bundle with Spin structures on another bundle.

Theorem 1.2.20. *Let B be a CW-complex and $E\to B$ be a rank n vector bundle. There is a natural one-to-one correspondence between the equivalence classes of* Pin^- *structures on E and the equivalence classes of* Spin *structures on $E\oplus\det E$.*

Analogously, there is a natural one-to-one correspondence between the equivalence classes of Pin^+ *structures on E and the equivalence classes of* Spin *structures on $E\oplus 3\cdot\det E$.*

Proof. The proofs for Pin^+ and Pin^- are formally the same, so we restrict ourselves to the Pin^- case. Let $\{g_{\alpha\beta}\}$ be a cocycle representing E with values in $\mathrm{O}(n)$. Then $E\oplus\det E$ is represented by

$$\{G_{\alpha\beta}\}=\left\{\begin{bmatrix} g_{\alpha\beta} & 0\\ 0 & \det g_{\alpha\beta}\end{bmatrix}\right\}.$$

Let $\{h_{\alpha\beta}\}$ be a cocycle representing a Pin^- structure on E that satisfy $\lambda(h_{\alpha\beta}) = g_{\alpha\beta}$, and define the $\mathrm{Pin}^-(n+1)$-valued functions

$$x_{\alpha\beta} = \begin{cases} 1, & \text{if } \det g_{\alpha\beta} = 1 \\ e_{n+1}, & \text{if } \det g_{\alpha\beta} = -1. \end{cases}$$

Then $\{H_{\alpha\beta}\} := \{h_{\alpha\beta} \cdot x_{\alpha\beta}\}$ is a family of Pin^--valued maps that satisfy $\lambda(H_{\alpha\beta}) = G_{\alpha\beta}$. Since $G_{\alpha\beta}$ take values in $\mathrm{SO}(n+1)$, the functions $H_{\alpha\beta}$ are actually take values in $\mathrm{Spin}(n+1)$. They are even a cocycle because $H_{\alpha\alpha} = \mathrm{id}$ and

$$\begin{aligned} H_{\alpha\beta}H_{\beta\gamma}H_{\gamma\alpha} &= h_{\alpha\beta} \cdot x_{\alpha\beta} \cdot h_{\beta\gamma} \cdot x_{\beta\gamma} \cdot h_{\gamma\alpha} \cdot x_{\gamma\alpha} \\ &= (-1)^{|h_{\gamma\alpha}|\cdot|x_{\beta\gamma}|+(|h_{\beta\gamma}|+|h_{\gamma\alpha}|)\cdot|x_{\alpha\beta}|} h_{\alpha\beta} \cdot h_{\beta\gamma} \cdot h_{\gamma\alpha} \cdot x_{\alpha\beta} \cdot x_{\beta\gamma} \cdot x_{\gamma\alpha} \\ &= (-1)^{|x_{\gamma\alpha}|\cdot|x_{\beta\gamma}|+(|x_{\beta\gamma}|+|x_{\gamma\alpha}|)\cdot|x_{\alpha\beta}|} h_{\alpha\beta} \cdot h_{\beta\gamma} \cdot h_{\gamma\alpha} \cdot x_{\alpha\beta} \cdot x_{\beta\gamma} \cdot x_{\gamma\alpha} \\ &= (-1)^{|x_{\gamma\alpha}|\cdot|x_{\beta\gamma}|+(|x_{\beta\gamma}|+|x_{\gamma\alpha}|)\cdot|x_{\alpha\beta}|} x_{\alpha\beta} \cdot x_{\beta\gamma} \cdot x_{\gamma\alpha} \\ &= 1. \end{aligned}$$

The last equality follows because only none or two of the three function can be of odd degree since $\{\det g_{\alpha\beta}\}$ is a cocycle. Indeed, if all functions are of the same degree the last equation holds trivially. Otherwise, we may assume without loss of generality that $x_{\alpha\beta} = x_{\beta\gamma} = e_{n+1}$. Then $x_{\gamma\alpha} = 1$ and the last equality reduces to

$$(-1)^{|x_{\alpha\beta}|\cdot|x_{\beta\gamma}|} x_{\alpha\beta} \cdot x_{\beta\gamma} = (-1)(-1) = 1.$$

Thus, $\{H_{\alpha\beta}\}$ defines a Spin structure on $E \oplus \det E$.

Conversely, any Spin structure of $E \oplus \det E$ can be represented by a cocycle $\{H_{\alpha\beta}\}$ with $\lambda(H_{\alpha\beta}) = G_{\alpha\beta}$; therefore, $H_{\alpha\beta} = h_{\alpha\beta}x_{\alpha\beta}$. We claim that $h_{\alpha\alpha} = 1$ and that $\{h_{\alpha\beta}\}$ satisfies the cocycle condition, and defines therefore a Pin^- structure on E. The first assertion is obvious, the second one follows if we do the previous calculation backwards:

$$\begin{aligned} 1 &= H_{\alpha\beta}H_{\beta\gamma}H_{\gamma\alpha} \\ &= h_{\alpha\beta} \cdot x_{\alpha\beta} \cdot h_{\beta\gamma} \cdot x_{\beta\gamma} \cdot h_{\gamma\alpha} \cdot x_{\gamma\alpha} \\ &= h_{\alpha\beta} \cdot h_{\beta\gamma} \cdot h_{\gamma\alpha} \cdot (-1)^{|h_{\gamma\alpha}|\cdot|x_{\beta\gamma}|+(|h_{\beta\gamma}|+|h_{\gamma\alpha}|)\cdot|x_{\alpha\beta}|} \cdot x_{\alpha\beta} \cdot x_{\beta\gamma} \cdot x_{\gamma\alpha} \\ &= h_{\alpha\beta} \cdot h_{\beta\gamma} \cdot h_{\gamma\alpha} \cdot (-1)^{|x_{\gamma\alpha}|\cdot|x_{\beta\gamma}|+(|x_{\beta\gamma}|+|x_{\gamma\alpha}|)\cdot|x_{\alpha\beta}|} \cdot x_{\alpha\beta} \cdot x_{\beta\gamma} \cdot x_{\gamma\alpha} \\ &= h_{\alpha\beta} \cdot h_{\beta\gamma} \cdot h_{\gamma\alpha}. \end{aligned}$$

The described assignments are inverse to each other. Furthermore, they are equivariant with respect to the $H^1(B;\mathbb{Z}_2)$-action on the cocycles. □

The previous theorem allows to assign to each Pin^- manifold of dimension n a Spin manifold of dimension $n+1$ in a functorial way. This construction will be useful for relating Pin^- bordism with Spin bordism.

Lemma 1.2.21. *For a manifold M let $M^{or} = O(\det TM) \xrightarrow{p} M$ be its orientation covering. Let $\mathbb{Z}_2$ act on $S^1 \subseteq \mathbb{C}$ by complex conjugation and denote the associated S^1 fibre bundle $M^{or} \times_{\mathbb{Z}_2} S^1$ over M with $\mathcal{S}(M)$. Then the following assertions hold:*

1. $\mathcal{S}(M) \xrightarrow{p} M$ *has two global section* $\sigma_{\pm 1}\colon m \mapsto [\tilde{m}, \pm 1]$, *where* $\tilde{m}$ *is some element in the fibre of* m *in* $\tilde{M}$.
2. $\nu(\sigma_{\pm 1}) \cong \det TM$.
3. $T\mathcal{S}(M) \cong p^*(TM \oplus \det TM)$.
4. *There is an embedding* $\exp\colon \det TM \to \mathcal{S}(M) \setminus \operatorname{im} \sigma_{-1}$.

Proof. The fibre bundle $\mathcal{S}$(M) can be constructed alternatively as follows. Let $\{(U_\alpha, \Phi_\alpha)\}$ be a set of local trivialisation of TM such that the trivialisation domains cover M and the trivialisation maps yield transition functions with values in $\mathrm{O}(n)$. Then

$$\mathcal{S}(M) = \left(\bigsqcup_\alpha U_\alpha \times S^1\right) \Big/ \sim,$$
$$(\alpha, x, \zeta) \sim (b, y, \omega) \Leftrightarrow x = y \text{ and } \zeta = g_{\alpha\beta}\omega$$

because both of these S^1-fibre bundles have the same trivialisation domains and the same transition functions.

The first assertion is obvious because ± 1 lies in the stabiliser of the $\mathbb{Z}_2$ action. Using the alternative description of $\mathcal{S}$(M), we observe for the normal bundle of $\sigma_{\pm 1}\colon M \hookrightarrow \mathcal{S}$(M) the identity

$$\nu(\sigma_{\pm 1})|_{U_\alpha} = \nu\left(U_\alpha \times \{\pm 1\} \hookrightarrow U_\alpha \times S^1\right).$$

So, the isomorphism between $\nu(\sigma_{\pm 1})$ and $\det TM$ is locally given by

$$U_\alpha = \det TM|_{U_\alpha} \to \nu\left(U_\alpha \times \{1\} \hookrightarrow U_\alpha \times S^1\right) = \nu(\sigma_{\pm 1})|_{U_\alpha}$$
$$(u, \lambda) \mapsto [t \mapsto \exp(\mathrm{i}\lambda t)].$$

In order to prove the third assertion, we consider the well-defined bundle map

$$T\mathcal{S}(M) \to TM \oplus \nu(\sigma_{\pm 1}),$$
$$[\tilde{\gamma}_1, \gamma_2] \mapsto \left([p \circ \gamma_1], \left[\gamma_2(0)^{-1} \cdot \gamma_2\right]\right).$$

Since it covers $p\colon \mathcal{S}(\mathrm{M}) \to M$ and is fibre-wise an isomorphism, we conclude

$$T\mathcal{S}(M) \cong p^*(TM \oplus \nu(\sigma_{\pm 1})) \cong p^*(TM \oplus \det TM).$$

For the last assertion, fix an odd, monotonically increasing diffeomorphism $\phi\colon \mathbb{R} \to (-\pi; \pi)$. Then

$$\exp\colon \det TM = M^{or} \times_{\mathbb{Z}_2} \mathbb{R} \to M^{or} \times S^1 = \mathcal{S}(M),$$
$$[\tilde{m}, v] \mapsto [\tilde{m}, \exp(\mathrm{i}\phi(v))]$$

is a well defined embedding. Its image complements the image of the constant (-1)-valued section σ_{-1}. □

Definition 1.2.22. Let M be a Pin^- manifold with a fixed Pin^- structure. By Theorem 1.2.20 this Pin^- structure corresponds to a unique Spin structure on $TM \oplus \det TM$. We call $\mathcal{S}$(M) endowed with the pullback of this Spin structure the *Spinification of* M.

We close this section by considering of $\mathrm{Pin}^{\pm}$ structures on connected sums. The key observation is that two $\mathrm{Pin}^{\pm}$ structures can be glued together, if they are compatible. The next lemma gives a more precise description.

Lemma 1.2.23. *For $j \in \{1,2\}$ let $P_j \xrightarrow{\rho_j} O_j$ be two $\mathrm{Pin}^{\pm}$ structures over the bundles $O_j \to X_j$. Let $U_j \subseteq X_j$ be two open subsets and $\varphi\colon U_1 \to U_2$ be an isometric diffeomorphism such that $\mathrm{O}(\varphi)^* P_2|_{U_1}$ is equivalent to $P_1|_{U_1}$.*

Then there is a unique $\mathrm{Pin}^{\pm}$ structure P over the bundle $O_1 \cup_{\mathrm{O}(\varphi)} O_2 \to X_1 \cup_{\varphi} X_2$ such that $P|_{X_j} = P_j$.

Proof. Denote the composition of the upper horizontal line of

$$\begin{array}{ccccc} P_1|_{U_1} & \longrightarrow & \mathrm{O}(\varphi)^* P_2|_{U_2} & \longrightarrow & P_2|_{U_2} \\ \downarrow & & \downarrow & & \downarrow \\ O_1|_{U_1} & \xrightarrow{\mathrm{id}} & O_1|_{U_1} & \xrightarrow{\mathrm{O}(\varphi)} & O_2|_{U_2} \end{array}$$

with θ and define

$$P := P_1 \cup_\theta P_2 \xrightarrow{\rho_1 \cup_\theta \rho_2} O_1 \cup_{\mathrm{O}(\varphi)} O_2.$$

Since θ is equivariant the space $P_1 \cup_\theta P_2$ is a principal $\mathrm{Pin}^{\pm}$-bundle and the map $\rho_1 \cup_\theta \rho_2$ is equivariant as well. Thus, P is indeed a $\mathrm{Pin}^{\pm}$ structure.

Since the canonical inclusions $P_j \hookrightarrow P_1 \cup_\theta P_2$ are equivariant, the second statement follows. □

Let us recall the definition of the connected sum of two manifolds.

Definition 1.2.24. Consider the diffeomorphism

$$\begin{aligned} \psi\colon\ (D^n)^\circ \setminus \{0\} &\to (D^n)^\circ \setminus \{0\}, \\ x &\mapsto \frac{1 - ||x||}{||x||} \cdot x. \end{aligned}$$

For two smooth n-dimensional manifolds M_i and two charts $\varphi_i\colon U_i \to (D^n)^\circ$ we define the *connected sum* via

$$M_0 \# M_1 := M_0 \cup_{\varphi_1^{-1} \circ \psi \circ \varphi} M_1.$$

We refer to the U_i as *gluing domain.*

It is easily verified that the connected sums of two manifolds is again a smooth manifold. However, its diffeomorphism class may very well depend on the chosen chart; for example, $\mathbb{C}P^2 \# \mathbb{C}P^2$ is not homotopy equivalent to $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$, because they have different signatures. But this shall not bother us because their $\mathrm{Pin}^{\pm}$ bordism class will be the same. Therefore, we will omit the chosen charts in the notation.

Observe that the diffeomorphism ψ maps the upper half-plane $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ to itself. Therefore, the construction above can be applied to boundary points as well. We denote the *connected sum along boundaries* with $M_0 \#_b M_1$.

Theorem 1.2.25 (Pin structures on connected sums). *Let M_1 and M_2 be two* $\mathrm{Pin}^\pm$ *manifolds of dimension $n \neq 2$ with fixed* $\mathrm{Pin}^\pm$ *structure. Then there is a unique* $\mathrm{Pin}^\pm$ *structure on the connected sum $M_1 \# M_2$ such that the inclusions $M_j \setminus \varphi_j^{-1}(\frac{1}{3}D^n) \hookrightarrow M_1 \# M_2$ are* $\mathrm{Pin}^\pm$ *structure preserving.*

If $n = 2$, then this statement also holds for Pin^- *manifolds.*

The result carries over to $M_1 \#_b M_2$ for all $\mathrm{Pin}^\pm$ *manifolds with non-empty boundaries, even to* Pin^+ *manifolds of dimension* 2.

Proof. Choose Riemannian metrics on M_1, M_2, and $M_1 \# M_2$ such that the inclusions $M_j \setminus \varphi_j^{-1}(\frac{1}{3}D^n) \hookrightarrow M_1 \# M_2$ become isometric.

If we set $\varphi := \varphi_2^{-1} \circ \varphi_1$ and use the open subsets $U_j := \varphi_j^{-1}\left((D^n)^\circ\right) \setminus \varphi_j^{-1}(\frac{1}{3}D^n)$, almost all requirements of Lemma 1.2.23 are satisfied. We only need to verify that φ is $\mathrm{Pin}^\pm$ structure preserving.

To this end, recall that the set of equivalence classes of $\mathrm{Pin}^\pm$ structures on U_1 is parametrised by $H^1(U_1; \mathbb{Z}_2) \cong H^1(S^{n-1}; \mathbb{Z}_2) \cong 0$ as long as $n \neq 2$. Thus, in this case φ is automatically $\mathrm{Pin}^\pm$ structure preserving and the first part follows from Lemma 1.2.23.

Now, let $n = 2$. As in the first part it only remains to show that φ preserves the $\mathrm{Pin}^\pm$ structures. Since $U_1 \approx (D^2)^\circ \setminus \frac{1}{3}D^2 \approx \frac{1}{2}\left(S^1 \times \left(\frac{2}{3}, 2\right)\right)$, a $\mathrm{Pin}^\pm$ structure on U_1 is uniquely determined by its restriction to $\varphi^{-1}(\frac{1}{2}S^1)$. But the Pin^- structures on $\varphi_j^{-1}(\frac{1}{2}S^1)$ are restrictions from the Pin^- structures on $\varphi^{-1}\left(\frac{1}{2}D^2\right)$, so the manifolds $\varphi_j^{-1}(S^1)$ are Pin^- boundaries. After we have introduced the Pin^- bordism groups, we will see that this property determines the Pin^- structure uniquely up to equivalence. Thus, φ is Pin^- structure preserving.

For $M_1 \#_b M_2$ the goes analogously. Here, we have no case distinction because the first cohomology of $U_j \approx \frac{1}{2}\left(D^n \times \left(\frac{2}{3}, 2\right)\right)$ vanishes. □

Note that the proof of the second part cannot be adapted to the Pin^+ case, because the circle with the 'bad' Spin structure bounds at the Möbius-strip [KT90b]. In any case, the correctness of the second part for the Pin^+ case is only of minor importance for this thesis because we apply this theorem only to Pin^- manifolds.

1.3. The Pin bordism groups

This section presents the key objects of the entire thesis. We will use the concepts from the previous section to define the Pin^- bordism groups and prove elementary facts about them. Although the proofs presented here generalise often to many other topological groups, we will only discuss Pin^- bordism. The reader is referred to Appendix B for a summary through bordism theory.

Definition 1.3.1 (singular Pin manifolds). Let X be a topological space. A *singular Pin⁻ manifold* is a triple $(M, P \xrightarrow{\rho} \mathrm{O}(TM), f)$ consisting of a smooth compact Pin^- manifold M with a fixed Pin^- structure (P, ρ) and a continuous map $f\colon M \to X$. We omit the Pin^- structure and just write (M, f) if it does not lead to confusions. In this case, the singular Pin^- manifold with inverse Pin^- structure will be denoted with $(\bar{M}, f)$.

Definition 1.3.2. Two closed singular Pin^- manifolds (M_i, f_i) of dimension n are called *Pin^- bordant*, if there is a $n+1$ dimensional singular Pin^- manifold with boundary (W, F) and a Pin^- structure preserving diffeomorphism $\varphi\colon \partial W \to M_0 \sqcup \bar{M}_1$ satisfying $(f_0 \sqcup f_1) \circ \varphi = F|_{\partial W}$.

The singular Pin^- manifold (W, F) is called a *Pin^- bordism* between (M_0, f_0) and (M_1, f_1).

Lemma 1.3.3. *Being* Pin^- *bordant is an equivalence relation.*

Proof. Reflexivity is proven in Example 1.2.18.

Symmetry follows from the following observation. If (W, F) is a Pin^- bordism between (M_0, f_0) and (M_1, f_1), then $(\bar{W}, F)$ is a Pin^- bordism between (M_1, f_1) and (M_0, f_0).

Transitivity is a little bit harder. Let (V, F_{01}) be a bordism between (M_0, f_0) and (M_1, f_1), and (W, F_{12}) a bordism between (M_1, f_1) and (M_2, f_2). To keep notation simple, we assume $\partial V = M_0 \sqcup \bar{M}_1$ and $\partial W = M_1 \sqcup \bar{M}_2$. By the collar neighbourhood theorem, there are open neighbourhoods $U_1 \subseteq V$ and $U_2 \subseteq W$ of M_1 such that $U_i \approx M_1 \times [0, 1)$ via diffeomorphisms mapping M_1 identically to itself. An argument analogous to the one in Example 1.2.18 shows that $M_1 \times (-1, 0] \cup M_1 \times [0, 1) = M \times (-1, 1)$ possesses a unique Pin^- structure which restricts on $M_1 \times [0, 1)$ and $M \times (-1, 0]$ to the given ones. So, $(V \cup_{M_1} W, F_{01} \cup F_{12})$ is a singular Pin^- manifold giving a Pin^- bordism between (M_0, f_0) and (M_1, f_1). □

More generally, two not necessarily closed n-dimensional singular Pin^- manifolds (M_i, f_i) are Pin^- bordant if there is a Pin^- bordism (U, f) between $(\partial M_0, f_0|\partial M_0)$ and $(\partial M_1, f_1|_{\partial M_1})$, such that the closed singular Pin^- manifold $(M_0 \cup_{M_0} \bar{U} \cup_{\partial M_1} M_1, f_0 \cup f_1 \cup f_2)$, obtained by proper boundary modifications, is a Pin^- boundary, i.e. it is Pin^- bordant to the empty set. One can show analogously that the general bordism relation is an equivalence relation.

Definition 1.3.4. Let (X, A) be a pair of topological spaces. The *n-th Pin^- bordism group* of (X, A) is defined by

$$\Omega_n^{\mathrm{Pin}^-}(X, A) := \frac{\{(M, f) \mid (M, f) \text{ sing. } \mathrm{Pin}^- \text{ manifold, } f(\partial M) \subseteq A\}}{\text{bordism}}.$$

If A is empty, we will write $\Omega_n^{\mathrm{Pin}^-}(X)$ instead of $\Omega_n^{\mathrm{Pin}^-}(X, \emptyset)$. Note that the condition $A = \emptyset$ forces the singular manifolds to be closed.

Lemma 1.3.5. $\Omega_n^{\mathrm{Pin}^-}(X, A)$ *is an abelian group. Addition is induced by disjoint union, the neutral element is given by an arbitrary singular* Pin^- *manifold that which is a* Pin^- *boundary, and the inverse element is given by the same singular manifold but with inverse* Pin^- *structure.*

Proof. The addition $[M_0, f_0] + [M_1, f_1] := [M_0 \sqcup M_1, f_0 \sqcup f_1]$ is well-defined. If (N_0, g_0) and (N_1, g_1) are other representatives, then the disjoint union of the relating bordisms (W_i, F_i) gives a bordism between $(M_0 \sqcup M_1, f_0 \sqcup f_1)$ and $(N_1 \sqcup N_2, g_1 \sqcup g_2)$.

Associativity and commutativity follow then from associativity and commutativity of disjoint union.

If (S, f) is a singular Pin^- boundary of (B, F), then a bordism between $(S \sqcup M, f \sqcup f_1)$ and (M, f_1) is given by $(B \sqcup M \times [0, 1], f \sqcup f_1 \circ \mathrm{pr}_1)$. By choosing $B = \emptyset$ this bordism shows further that $(M \sqcup \bar{M}, f_1 \sqcup f_1)$ is a Pin^- boundary, and therefore $-[M, f_1] = [M, f_1]$. □

Note that our description of Pin^- bordism is intrinsic because it does not rely on the choice of some embedding. However, in order to use the famous Pontrjagin-Thom construction it is useful to have a definition at hand that uses the notion of normal bundles.

By Lemma 1.2.14 we know that a Pin^- structure on a manifold M induces a unique Pin^+ structure on $\nu(f\colon M \hookrightarrow \mathbb{R}^N)$, because $M \times \mathbb{R}^N$ inherits a canonical Pin^- structure from $T\mathbb{R}^N$ by restriction. Therefore, we get a stable Pin^+ structure on $\nu(f\colon M \hookrightarrow \mathbb{R}^N)$, and our definition agrees with the one usually given in the literature, like [DK01] or [Sto15].

The Pontrjagin-Thom construction gives an isomorphism

$$\Omega_n^{\mathrm{Pin}^-}(X, A) \cong \lim_{k\to\infty} \pi_{k+n}(M\mathrm{Pin}_k^+ \wedge X_+/A_+) = H_n(X, A; \mathbf{MPin}^+).$$

For the proof see [DK01], [Swi02] or [Sto15].

With a little faith one sees that the Pontrjagin-Thom construction is a natural transformation between the functors $\Omega_*^{\mathrm{Pin}^-}(\cdot,\cdot)$ and $H_*(\cdot,\cdot;\mathbf{MPin}^+)$. Thus, Pin^- bordism is a generalised homology theory [Swi02, Chapter 8].

The connecting homomorphism $\partial\colon \Omega_n^{\mathrm{Pin}^-}(X, A) \to \Omega_{n-1}^{\mathrm{Pin}^-}(A)$ is explicitly given by restricting a singular manifold to its boundary, more precisely $\partial[M, f] = [\partial M, f|_{\partial M}]$.

Theorem 1.3.6. *The addition on $\Omega_*^{\mathrm{Pin}^-}(\mathrm{pt})$ is also induced by the connected sum of two manifolds.*

Proof. Let M_1 and M_2 be two closed Pin^- manifolds. We have to show that $M_1\#M_2$, endowed with the Pin^- structure constructed in Theorem 1.2.25, and $M_1 \sqcup M_2$ are Pin^- bordant. But a bordism between $M_1\#M_2$ and $M_1 \sqcup M_2$ is given by $(M_1 \times I)\#_b(M_2 \times I)$. Indeed, as we have seen in Example 1.2.18 the Pin^- structures on M_i induce unique Pin^- structures on $M \times I$ turning the inclusion $M_i = M_i \times \{0\} \hookrightarrow M \times I$ into Pin^- structure preserving embeddings. This gives, by Theorem 1.2.25, a unique Pin^- structure on $(M_1 \times I)\#_b(M_2 \times I)$, where we glued along boundary points lying in $M_i \times \{0\}$. The identification $M_i = M_i \times \{0\}$ gives a Pin^- structure preserving inclusion $M_1\#M_2 \hookrightarrow (M_1 \times I)\#_b(M_2 \times I)$. This follows from Example 1.2.18 combined with Theorem 1.2.25.

Since the connected sum construction leaves the Pin^- structure on $M_i \times \{1\}$ unchanged, we conclude from Example 1.2.18

$$\partial\left((M_1 \times I)\#_b(M_2 \times I)\right) = (M_1\#M_2) \sqcup (\bar{M}_1 \sqcup \bar{M}_2)$$

Therefore, $(M_1 \times I)\#_b(M_2 \times I)$ serves indeed as Pin^- bordism between the connected sum and the disjoint union of two Pin^- manifolds. □

Since manifolds are locally contractible, every continuous function can be homotopied to a map that is constant on a neighbourhood of some point.

This allows to generalise the connected sum construction to singular manifolds, and we can extend the result of Theorem 1.3.6 to arbitrary path connected topological spaces.

Theorem 1.3.7. *Let M_1 and M_2 be two compact, connected* Pin^- *manifolds of the same dimension and $f_i\colon M_i \to X$ be continuous with images in the same path connected component.*

Then $[M_1, f_1] + [M_2, f_2] = [M_1 \# M_2, f_1' \# f_2']$, where $f_i' \simeq f_i$ are constant on the gluing domain and take the same value.

Proof. Let $f\colon \bar{D}^n \to X$ be continuous, $q < 1$, and be $g\colon q\bar{D}^n \to X$ an arbitrary continuous function that maps into the same path connection component as f. Since the (closed) disk $\bar{D}^n$ is contractible, f can be homotopied relative S^{n-1} to a continuous function $f'\colon \bar{D}^n \to X$ with $f'|_{q\bar{D}^n} = g$.

Thus, we can homotopy f_i into continuous maps f_i' that are constant on the glueing domain, and, because f_1 and f_2 take values in the same path connected component, we can arrange that f_1' equals f_2' on the gluing domain.

Since $[M_i, f_i] = [M_i, f_i']$, it remains to construct a Pin^- bordism between $(M_1 \# M_2, f_1' \# f_2')$ and $(\bar{M}_1 \sqcup \bar{M}_2, f_1 \sqcup f_2)$, but such a bordism is, for example, given by $((M_1 \times I)\#_b(M_2 \times I), (f_1' \circ \mathrm{pr}_1)\#_b(f_2' \circ \mathrm{pr}_2))$ if we choose the same Pin^- structures as in the previous theorem. □

Corollary 1.3.8. *If X is path connected, then the addition on $\Omega_n^{\mathrm{Pin}^-}(X)$ is induced by the connected sum. In particular, every element can be represented by a connected* Pin^- *manifold.*

An immediate consequence is that class in $\Omega_n^{\mathrm{Pin}^-}(\mathrm{pt})$ can be represented by connected Pin^- manifolds. Combining this observation with the classification theorem for surfaces allows us to derive an important result for the structure of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{pt})$.

Theorem 1.3.9. $\Omega_2^{\mathrm{Pin}^-}(\mathrm{pt})$ *is cyclic. The generator is $[\mathbb{R}P^2]$ with an arbitrary* Pin^- *structure.*

Proof. With the help of the Mayer-Vietoris sequence we derive inductively the following isomorphisms

$$H^1(\#^k\mathbb{R}P^2; \mathbb{Z}_2) \xrightarrow[\cong]{\oplus \mathrm{incl}^*} H^1(\mathbb{R}P^2 \setminus (D^2 \sqcup D^2); \mathbb{Z}_2) \xleftarrow[\cong]{\oplus \mathrm{incl}^*} \bigoplus H^1(\mathbb{R}P^2; \mathbb{Z}_2) \cong \mathbb{Z}_2^k.$$

Since the first cohomology of the base space acts freely and transitively on the set of Pin^- structures, we conclude from this bijection that every Pin^- structure on $\#^k\mathbb{R}P^2$ can be obtained from the gluing construction as described in the proof of Theorem 1.2.25.

Every element $x \in \Omega_2^{\mathrm{Pin}^-}(\mathrm{pt})$ can be represented by a connected manifold. The classification theorem of surfaces states that every compact connected two dimensional manifold is diffeomorphic to a surface of the form $(\#^k\mathbb{R}P^2)\#(\#^l T^2)$. If $k = l = 0$, we define this expression to be S^2. However, since $\mathbb{R}P^2 \# T^2$ is diffeomorphic $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, we conclude from the equation

$$\begin{aligned} x &= \left[(\#^k \mathbb{R}P^2) \# (\#^l T^2)\right] \\ &= \left[(\#^k \mathbb{R}P^2) \# (\#^l T^2)\right] + \left[\#^l \mathbb{R}P^2\right] - \left[\#^l \mathbb{R}P^2\right] \\ &= \left[\#^{k+3l} \mathbb{R}P^2\right] - \left[\#^l \mathbb{R}P^2\right] \\ &= \left[(\#^{k+3l} \mathbb{R}P^2) \# \overline{(\#^l \mathbb{R}P^2)}\right] \end{aligned}$$

that it can be represented by a connected sum of real projective planes with some Pin^- structure. But we have deduced that every Pin^- structure on a connected sum of projective planes is obtained from the gluing construction. Thus, $\Omega_2^{\mathrm{Pin}^-}(\mathrm{pt})$ is generated by $[\mathbb{R}P^2]$ and is therefore cyclic. □

Remark 1.3.10. We will see later that $\Omega_2^{\mathrm{Pin}^-}(\mathrm{pt})$ contains eight elements, so it is isomorphic to $\mathbb{Z}_8$, see Corollary 1.5.2.

1.4. Relations between Pin and Spin bordism

The aim of this section is to determine the Pin^- bordism groups in terms of Spin bordism. The main result will be the existence of an isomorphism

$$\Omega_n^{\mathrm{Pin}^-}(X) \cong \Omega_{n+1}^{\mathrm{Spin}}(\mathbb{R}P^\infty \times X, X),$$

which we will use later to calculate the needed Pin^- bordism groups $\Omega_*^{\mathrm{Pin}^-}(\mathrm{pt})$.

The structure of this section can be summarised as follows. If γ_∞ denotes the universal line bundle and $\mathbb{R}^n_{\mathrm{Spin}} = E\mathrm{Spin}(n) \times_{\mathrm{Spin}(n)} \mathbb{R}^n$ denotes the universal $\mathrm{Spin}(n)$ vector bundle, then the exterior sum

$$\gamma_\infty \times \mathbb{R}^n_{\mathrm{Spin}} = \mathrm{pr}_1^* \gamma_\infty \oplus \mathrm{pr}_2^* \mathbb{R}^n_{\mathrm{Spin}} \to B\mathrm{O}(1) \times B\mathrm{Spin}(n)$$

has a Pin^+ structure, so there is a classifying map g_n for it. Stability give rise to a map $g\colon B\mathrm{O}(1) \times B\mathrm{Spin} \to B\mathrm{Pin}^+$, which will turn out to be a homotopy equivalence. The isometric vector bundle maps covering g_n give rise to a map between the Thom spectra $\mathbf{MO}(1) \wedge \mathbf{MSpin}$ and $\mathbf{MPin}^+$. Using an auxiliary bordism theory and the Pontrjagin Thom isomorphism we will derive geometrically that this map is a weak homotopy equivalence and by Whiteheads theorem for spectra it will be a homotopy equivalence. This result strengthens the result in [KT90, Lemma 6]. In this section, we will make use of the models $B\mathrm{O}(1) \times B\mathrm{Spin}(n)$ and $B(\mathrm{O}(1) \times \mathrm{Spin}(n))$ at different places because it makes the proofs easier. Although the spaces $B\mathrm{O}(1) \times B\mathrm{Spin}(n)$ and $B(\mathrm{O}(1) \times \mathrm{Spin}(n))$ are homotopy equivalent, we will nonetheless carefully distinguish these two models because sloppy identifications often lead to mistakes.

Let us now construct the homotopy equivalence between the classifying spaces. Recall that $B\iota_n\colon B\mathrm{Spin}(n) \to B\mathrm{Spin}(n{+}1)$ is unique up to homotopy and classify the universal $\mathrm{Spin}(n)$ bundle as a $\mathrm{Spin}(n+1)$ bundle. By replacing $B\mathrm{Spin}(n{+}1)$ with the mapping cylinder, we replace the map by a cofibration without changing the homotopy type of the target space. If we assume additionally that both spaces have the homotopy type of a CW-complex, then the mapping cylinder has the homotopy type of a CW-complex as well. Since we are free to choose our models, we may assume in the first place that $B\iota_n$ is an inclusion of a CW-subcomplex. Of course, the same holds for $B\mathrm{Pin}^+(n)$.

Since every line bundle possesses a $\mathrm{Pin}^+(1)$ structure, the universal line bundle γ_∞ has one. More generally, by Corollary 1.2.13, the bundle $\gamma_\infty \times \mathbb{R}^n_{\mathrm{Spin}} = \mathrm{pr}_1^*\gamma_\infty \oplus \mathrm{pr}_2^*\mathbb{R}^n_{\mathrm{Spin}}$ has a $\mathrm{Pin}^+(n+1)$ structure. We find therefore a classifying map $g_n\colon B\mathrm{O}(1) \times B\mathrm{Spin}(n) \to B\mathrm{Pin}^+(n+1)$. Stability gives the diagram

$$\begin{array}{ccc} B\mathrm{O}(1) \times B\mathrm{Spin}(n) & \xrightarrow{\;g_n\;} & B\mathrm{Pin}^+(n+1) \\ \Big\downarrow{\scriptstyle \mathrm{id}\times B\iota_n} & & \Big\downarrow{\scriptstyle B\iota_{n+1}} \\ B\mathrm{O}(1) \times B\mathrm{Spin}(n+1) & \xrightarrow{\;g_{n+1}\;} & B\mathrm{Pin}(n+2), \end{array}$$

which commutes up to homotopy. The vertical arrows are cofibrations, so we can homotopy g_{n+1} to a map making the diagram commutative. Since we are only interested in the homotopy type of g_{n+1}, we may assume that the diagram above already commutes in the first place.

If we set

$$B\mathrm{O}(1) \times B\mathrm{Spin} := \lim_{n\to\infty} B\mathrm{O}(1) \times B\mathrm{Spin}(n)$$

and

$$B\mathrm{Pin}^+ := \lim_{n\to\infty} B\mathrm{Pin}^+(n),$$

we derive the existence of a continuous map

$$g := \lim_{\to} g_n\colon B\mathrm{O}(1) \times B\mathrm{Spin} \to B\mathrm{Pin}^+.$$

The previous discussion applies to $B\mathrm{Pin}^+$ as well. Thus, we can assume that $B\mathrm{Pin}^+$ is a CW-complex such that the canonical inclusion $B\mathrm{Pin}^+(n) \hookrightarrow B\mathrm{Pin}^+$ is an inclusion of a subcomplex, which we will do for the rest of this section. We assume the same for $B\mathrm{O}(1) \times B\mathrm{Spin}$. We may even assume that all these spaces are countable CW-complexes because all homotopy groups have countably many elements [Hat02, p.359] and we are going to do that.

Classifying spaces identify the set of isomorphism classes of certain principal bundles over a given space with the set of homotopy classes of maps from this space into the corresponding classifying space. However, those are unpointed maps and the homotopies may be free, while homotopy groups are objects in the pointed world. The next lemma recalls the relation between the set of pointed homotopy classes and unpointed homotopy classes. A proof of the first part can be found in [DK01, Theorem 6.57], the second part is an easy observation.

Lemma 1.4.1. *Let (X, x_0), (Y, y_0), and (Z, z_0) be well pointed topological spaces,* i.e. *the base point is closed and its inclusion is a cofibration. Furthermore, Y and Z are assumed to be path connected.*

Then $\pi_1(Y)$ acts on $[X, x_0; Y, y_0]$ and the quotient set is precisely $[X, Y]$. Any continuous map $g\colon Y \to Z$ induces an equivariant map via postcomposition; more precisely

$$g_*([\gamma] . [f]) = \pi_1([\gamma]).g_*([f]).$$

Theorem 1.4.2. *For every $k \geq 0$, the map*

$$\pi_k(g)\colon \pi_k(B\mathrm{O}(1) \times B\mathrm{Spin}) \to \pi_k(B\mathrm{Pin}^+)$$

is an isomorphism.

Proof. For $k = 0$, there is nothing to prove because both spaces are path connected. Next, consider the case $k = 1$. If we choose models for $B\mathrm{O}(1)$ and $B\mathrm{Spin}$, which are countable CW-complexes, then on $B\mathrm{O}(1) \times B\mathrm{Spin}$ the product topology agrees with the CW topology, so we obtain for the fundamental group

$$\begin{aligned}\pi_1(B\mathrm{O}(1) \times B\mathrm{Spin}) &\cong \pi_1(B\mathrm{O}(1)) \oplus \pi_1(B\mathrm{Spin}) \\ &= \pi_1(B\mathrm{O}(1)) \oplus \lim_{n\to\infty} \pi_1(B\mathrm{Spin}(n)) \\ &= \left[S^1, B\mathrm{O}(1)\right] \oplus 0 \\ &= \left[S^1, B\mathrm{O}(1)\right] \cong \mathbb{Z}_2.\end{aligned}$$

Analogously, we verify

$$\pi_1(B\mathrm{Pin}^+) \cong \lim_{n\to\infty} \pi_1(B\mathrm{Pin}^+(n)) = \lim_{n\to\infty} \pi_0(\mathrm{Pin}^+(n)) = \mathbb{Z}_2.$$

Pick an element $[\alpha] \in \pi_1(B\mathrm{O}(1) \times B\mathrm{Spin})$ such that $\pi_1(g_n)([\alpha]) = [g_n \circ \alpha] = 0$. Then we derive

$$(g_n \circ \alpha)^* \mathbb{R}^{n+1}_{\mathrm{Pin}^+} = \alpha^*(\gamma_\infty \times \mathbb{R}^n_{\mathrm{Spin}}) = (\mathrm{pr}_1 \circ \alpha)^*(\gamma_\infty \times \varepsilon^n) \cong \varepsilon^{n+1}.$$

Since $[S^1, B\mathrm{O}(1)] = \mathbb{Z}_2$, the bundle $\mathrm{pr}_1 \circ \alpha$ is either isomorphic to the trivial bundle or the Möbius bundle. But the latter can be ruled out because it is not stably trivial since its first Stiefel-Whitney class does not vanish. Therefore, $\mathrm{pr}_1 \circ \alpha$ is nullhomotopic and so is α.

Since $\pi_1(B\mathrm{O}(1) \times B\mathrm{Spin})$ and $\pi_1(B\mathrm{Pin}^+)$ are abstractly isomorphic to $\mathbb{Z}_2$, we conclude also surjectivity from injectivity of $\pi_1(g_n)$.

For a generic $k \geq 2$, we make use of Lemma 1.4.1. Since $\pi_1(g)$ is an isomorphism, $\pi_k(g)$ will be an equivariant map, so it is enough to verify bijectivity of

$$g_*\colon \left[S^k, B\mathrm{O}(1) \times B\mathrm{Spin}\right] \to \left[S^k, B\mathrm{Pin}^+\right].$$

Take an element α that does not represent the constant class. Due to compactness of S^k, the image of α hits only finitely many cells in $B\mathrm{O}(1) \times B\mathrm{Spin}$ and lies therefore in $B\mathrm{O}(1) \times B\mathrm{Spin}(N)$ for a sufficiently large N. Thus, $\alpha^*(\gamma_\infty \times \mathbb{R}^n_{\mathrm{Spin}(N)})$ is not a stably-trivial bundle and so must not

$$(g_N \circ \alpha)^*(\mathbb{R}^{N+1}_{\mathrm{Pin}^+}) = \alpha^*(\gamma_\infty \times \mathbb{R}^N_{\mathrm{Spin}}).$$

Consequently, $[g \circ \alpha] = [g_N \circ \alpha]$ is not null-homotopic and injectivity is proven.

To prove surjectivity, we pick an arbitrary element $[\alpha] \in \left[S^k, B\mathrm{Pin}^+\right]$. Again, it can be represented by a map $\alpha\colon S^k \to B\mathrm{Pin}^+(N)$. Every vector bundle E over a sphere S^k is orientable because its first Stiefel-Whitney class takes values in the zero group. Thus, $\alpha^*\mathbb{R}^N_{\mathrm{Pin}^+}$ has a $\mathrm{Spin}(N)$ structure. The universal property of $B\mathrm{Spin}(N)$ guarantees the existence of a map β such that the diagram

$$\begin{array}{ccc} S^k & \xrightarrow{\alpha} & B\mathrm{Pin}^+(N) \\ & \searrow^{\beta} & \uparrow \\ & & B\mathrm{Spin}(N) \end{array}$$

commutes upto homotopy. With the help of β we derive an isomorphism

$$\begin{aligned} (\iota_n \circ \alpha)^*\mathbb{R}^{N+1}_{\mathrm{Pin}^+} &= \varepsilon \oplus \alpha^*\mathbb{R}^N_{\mathrm{Pin}^+} \cong \varepsilon \oplus \beta^*\mathbb{R}^N_{\mathrm{Spin}} \\ &= (\mathrm{const} \times \beta)^*(\gamma_\infty \times \mathbb{R}^N_{\mathrm{Spin}}) \\ &\cong (\mathrm{const} \times \beta)^* g^*_{N+1}\mathbb{R}^{N+1}_{\mathrm{Pin}^+}, \end{aligned}$$

which gives us a homotopy between between $\iota_n \circ \alpha$ and $g_n \circ (\mathrm{const} \times \beta)$. By passing to the limes we get, using relaxed notation, $\alpha = g \circ (\mathrm{const} \times \beta)$. We have verified surjectivity, and the theorem is therefore proven. □

From Whiteheads theorem, which states that every weak homotopy equivalence is actually a homotopy equivalence, we immediately deduce the next corollary.

Corollary 1.4.3. $g\colon B\mathrm{O}(1) \times B\mathrm{Spin} \to B\mathrm{Pin}^+$ *is a homotopy equivalence.*

Now, we are going to describe how the homotopy equivalence g give rise to a homotopy equivalence between $\mathbf{MO}(1) \wedge \mathbf{MSpin}$ and $\mathbf{MPin}^+$. In order to do this, we have to introduce some notation.

Definition 1.4.4. Let $E \to B$ be a vector bundle and g be some Riemannian metric on it. We have the *unit disk bundle*

$$D(E) := \{v \in E \,|\, g(v,v) \leq 1\}$$

and the *unit sphere bundle*

$$S(E) := \{v \in E \,|\, g(v,v) = 1\}.$$

The *Thom space* is defined by

$$\mathrm{Th}(E) := D(E)/S(E).$$

The point $[S(E)]$ is often denoted by ∞.

The homeomorphism class of the Thom space is actually independent of the Riemannian metric. Indeed, renormalising gives a homeomorphism between two Thom spaces that are constructed by using different metrics. Even better, in [tD08, p.533] there is a metric independent construction via mapping cylinders. Furthermore, it is easy to see that vector bundle maps induces maps between Thom spaces, where the bundle maps don't need to be isometries if we use the metric independent construction. Even better, if the base space is a CW-complex, so is the Thom space, see [Swi02, p.228]. If $B_1 \subseteq B$ is a subcomplex, then $\mathrm{Th}(E|_{B_1}) \subseteq \mathrm{Th}(E)$ is a subcomplex.

Definition 1.4.5. $\mathbf{MPin}^+$ is the *Thom spectrum* associated to the stable Pin^+ structure, meaning $\mathbf{MPin}^+_n = \mathrm{Th}(\mathbb{R}^n_{\mathrm{Spin}})$. In the same manner, we define $(\mathbf{MO}(1) \wedge \mathbf{MSpin})_n := \mathbf{MO}(1) \wedge \mathbf{MSpin}_{n-1} := \mathrm{Th}(\gamma_\infty \times \mathbb{R}^{n-1}_{\mathrm{Spin}})$ and $\mathbf{M}(\mathbf{O}(1) \times \mathbf{Spin})_n := \mathrm{Th}(\mathbb{R}^n_{\mathrm{Spin}})$.

A proof that this definition leads indeed to a spectrum can be found in [Swi02, 12.29].

Since $\mathrm{O}(n)$, and hence $\mathrm{Pin}^+(n)$ and $\mathrm{O}(1)\times\mathrm{Spin}(n)$, are cellular groups, the join construction, endowed with the weak topology, carries a CW-structure, see A.3.3. Using the join-models for the classifying spaces, the group homomorphisms

$$\begin{array}{ccc} \mathrm{O}(1) \times \mathrm{Spin}(n) & \xhookrightarrow{j_n} & \mathrm{Pin}^+(n+1) \\ \downarrow & & \downarrow \lambda \\ \mathrm{O}(n+1) & \xrightarrow{\mathrm{id}} & \mathrm{O}(n+1) \end{array}$$

induces a strictly commuting diagram

$$\begin{array}{ccc} B\mathrm{O}(1) \times B\mathrm{Spin}(n) & \xhookrightarrow{Bj_n} & B\mathrm{Pin}^+(n+1) \\ \downarrow & & \downarrow B\lambda \\ B\mathrm{O}(n+1) & \xrightarrow{\mathrm{id}} & B\mathrm{O}(n+1) \end{array}$$

covered by strictly commuting isometric vector bundle maps

$$\begin{array}{ccc} \mathbb{R}^{n+1}_{\mathrm{O}(1)\times\mathrm{Spin}} & \xrightarrow{F_{n+1}} & \mathbb{R}^{n+1}_{\mathrm{Pin}^+} \\ \downarrow & & \downarrow \\ \mathbb{R}^{n+1}_{\mathrm{O}} & \xrightarrow{\mathrm{id}} & \mathbb{R}^{n+1}_{\mathrm{O}}. \end{array}$$

Here, j_n denotes the unique inclusion defined by $(-1, g) \mapsto e_0 \cdot g$ and F_n the isometric vector bundle map induced from Ej_n on the Borel construction, in terms of diagrams

$$\begin{array}{ccc} E\left(\mathrm{O}(1) \times \mathrm{Spin}(n)\right) \times_{\mathrm{O}(1)\times\mathrm{Spin}(n)} \mathbb{R}^{n+1} & \xrightarrow{Ej_n \times \mathrm{id}} & E\mathrm{Pin}^+(n+1) \times_{\mathrm{Pin}^+(n+1)} \mathbb{R}^{n+1} \\ \| & & \| \\ \mathbb{R}^{n+1}_{\mathrm{O}(1)\times\mathrm{Spin}} & \xrightarrow{F_{n+1}} & \mathbb{R}^{n+1}_{\mathrm{Pin}^+} \end{array}$$

All of these maps commute with the maps that arise from the inclusions of subgroups, so the are even stably commutative. We also have isometric classifying vector bundle maps

$$\begin{array}{ccc} \tau_\infty \times \mathbb{R}^n_{\mathrm{Spin}} & \longrightarrow & \mathbb{R}^{n+1}_{\mathrm{O}(1)\times\mathrm{Spin}(n)} \\ \downarrow & & \downarrow \\ B\mathrm{O}(1) \times B\mathrm{Spin}(n) & \xrightarrow{\simeq} & B(\mathrm{O}(1) \times \mathrm{Spin}(n)) \end{array}$$

that are also stably commutative. These maps induce maps on the corresponding Thom spaces, and by stability they give rise to maps between the Thom spectra

$$\mathbf{MO}(1) \wedge \mathbf{MSpin} \to \mathbf{M}(\mathbf{O}(1) \times \mathbf{Spin})$$

and

$$\mathrm{Th}(F)\colon \mathbf{M}(\mathbf{O}(1) \times \mathbf{Spin}) \to \mathbf{MPin}^+.$$

The first map is a homotopy equivalence because we also have a classifying vector bundle map from $\gamma_\infty \times \mathbb{R}^n_{\mathrm{Spin}}$ to $\mathbb{R}^{n+1}_{\mathrm{O}(1)\times\mathrm{Spin}}$. Since both bundles are universal, the composition of the classifying vector bundle maps are homotopic to the identity through an homotopy of isometric vector bundle maps. To prove that $\mathrm{Th}(F)$ is a homotopy equivalence, it will be convenient to consider an auxiliary bordism theory.

Definition 1.4.6. We define Ω^τ_n to be the set of bordism classes of all closed n-dimensional manifolds M that carry a Spin structure on $TM \oplus \det TM$.

By Theorem 1.2.20 the groups Ω^τ_n and $\Omega^{\mathrm{Pin}^-}_n$ are canonically isomorphic.

In order to determine the Thom-spectrum of the bordism theory Ω^τ_* we describe the defining condition in terms of stable normal bundles.

Lemma 1.4.7. *A manifold M has a* Spin *structure on M if and only if its stable normal bundle $\nu = (\nu_N)_N$ decomposes into a line bundle l, which will be automatically isomorphic to* $\det \nu$*, and a stable Spin bundle ξ.*

The set of equivalence classes of Spin *structures on $TM \oplus \det TM$ and the set of equivalence classes of* Spin *structures on ξ are in natural one-to-one correspondence.*

Proof. "$\Rightarrow$" After enlarging the ambient space, any two embeddings eventually become isotopic [Swi02, p.223]. Thus, the stable normal bundle depends only on M and not on the embedding. Let $\mathcal{S}(M) = M^{or} \times_{\mathbb{Z}_2} S^1$ be the S^1-fibre bundle over M constructed in Lemma 1.2.21. From the definition of M follows that $\mathcal{S}(M)$ has a Spin structure, and so its stable normal bundle ξ does. Since we can embed M into $\mathcal{S}(M)$ via the constant 1-section, we obtain for the stable normal bundle the decomposition

$$\begin{aligned} \nu(M \hookrightarrow \mathbb{R}^N) &= \nu(M \hookrightarrow \mathcal{S}(M)) \oplus \nu(\mathcal{S}(M) \hookrightarrow \mathbb{R}^N)|_M \\ &\cong \det TM \oplus \nu(\mathcal{S}(M) \hookrightarrow \mathbb{R}^N)|_M \\ &= \det \nu(M \hookrightarrow \mathbb{R}^N) \oplus \xi_N. \end{aligned}$$

"$\Leftarrow$" Let $\nu \cong \det\nu \oplus \xi$ such a decomposition for M and n the dimension of M. Then, for a sufficient large K and $r_K := \mathrm{rk}(\nu_K)$, we have a decomposition

$$\begin{aligned} \varepsilon^{r_K+n} &\cong \nu_K \oplus TM \cong \det\nu_K \oplus \xi_K \oplus TM \\ &= \xi_K \oplus \det\nu_K \oplus TM \\ &= \xi_K \oplus \det TM \oplus TM. \end{aligned}$$

Since ξ_K has a Spin structure, so does $\det TM \oplus TM$.

The statement about the one-to-one correspondence follows from the decomposition

$$\varepsilon^{r_K+n} \cong \xi_K \oplus \det TM \oplus TM$$

and the Two-out-of-Three-Lemma 1.2.14. $\square$

Corollary 1.4.8. $\Omega_n^\tau \cong \pi_n(\mathbf{M}(\mathbf{O}(1) \times \mathbf{Spin})) = \pi_n(\mathbf{MO}(1) \wedge \mathbf{MSpin})$.

Proof. This isomorphism follows from the Pontrjagin-Thom isomorphism and the previous lemma because the universal $\mathrm{O}(1) \times \mathrm{Spin}(N)$ vector bundle is given by

$$\mathbb{R}^{N+1}_{\mathrm{O}(1)\times\mathrm{Spin}} \simeq \gamma_\infty \times \mathbb{R}^N_{\mathrm{Spin}} \to B\mathrm{O}(1) \times B\mathrm{Spin}(N) \simeq B(\mathrm{O}(1) \times \mathrm{Spin}(N)).$$

Thus, we obtain for the Thom spaces the identity

$$\mathrm{Th}(\mathbb{R}^{N+1}_{\mathrm{O}(1)\times\mathrm{Spin}}) \simeq \mathrm{Th}(\gamma_\infty \times \mathbb{R}^N_{\mathrm{Spin}}) = \mathrm{Th}(\gamma_\infty) \wedge \mathrm{Th}(\mathbb{R}^N_{\mathrm{Spin}}),$$

and for the Thom spectra $\mathbf{M}(\mathbf{O}(1) \times \mathbf{Spin}) = \mathbf{MO}(1) \wedge \mathbf{MSpin}$. $\square$

The next Theorem presents a geometric interpretation of $\mathrm{Th}(F)$ via the Pontrjagin-Thom construction.

Theorem 1.4.9. *The diagram*

$$\begin{array}{ccc} \pi_n(\mathbf{M}(\mathbf{O}(1) \times \mathbf{Spin})) & \xrightarrow{\mathrm{Th}(F)_*} & \pi_n(\mathbf{MPin}^+) \\ \mathrm{PT}\uparrow\cong & & \mathrm{PT}\uparrow\cong \\ \Omega_n^\tau & \xrightarrow[{[M]\mapsto[M]}]{} & \Omega_n^{\mathrm{Pin}^-} \end{array}$$

commutes. The vertical maps are the Pontrjagin-Thom isomorphisms.

Proof. The statement follows directly from the commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n_{\mathrm{O}(1)\times\mathrm{Spin}} & \xrightarrow{F_n} & \mathbb{R}^n_{\mathrm{Pin}^+} \\ \downarrow & & \downarrow \\ \mathbb{R}^n_{\mathrm{O}} & \xrightarrow{\mathrm{id}} & \mathbb{R}^n_{\mathrm{O}}. \end{array}$$

Indeed, let M be a closed manifold with a fixed Spin structure on $TM \oplus \det TM$. We denote with ν the normal bundle of M induced by some embedding and also a classifying vector bundle map $\nu \to \mathbb{R}^N_{\mathrm{O}}$ that has a lift $\tilde{\nu}\colon \nu \to \mathbb{R}^N_{\mathrm{O}(1)\times\mathrm{Spin}}$ for the uniquely determined $\mathrm{O}(1) \times \mathrm{Spin}(N-1)$ reduction of the normal bundle that corresponds to the chosen Spin structure on $TM \oplus \det TM$. Since F_N is the vector bundle map that classify

$\mathbb{R}^N_{\mathrm{O}(1)\times\mathrm{Spin}}$ as $\mathrm{Pin}^+(N)$ vector bundle, the composition $F_N \circ \tilde{\nu}$ is the lift for the $\mathrm{O}(1) \times \mathrm{Spin}(N-1)$ reduction considered as Pin^+ reduction of ν. This Pin^+ structure on the normal bundle induces a unique Pin^- structure on TM by Theorem 1.2.20. By the uniqueness theorems 1.2.20 and 1.2.14, this Pin^- structure agrees with the Pin^- induced from the given Spin structure on $TM \oplus \det TM$ because both structures together with the $\mathrm{O}(1) \oplus \mathrm{Spin}(N-1)$ reduce to the trivial structure on the Whitney sum. □

Since the lower horizontal homomorphism is a bijection by Theorem 1.2.20, the upper horizontal morphism must be a bijection, too. We conclude that $\mathrm{Th}(F)$ is a weak homotopy equivalence and by Whiteheads Theorem for spectra it must be a homotopy equivalence.

Now, observe that the composition of the homotopy equivalence between $B\mathrm{O}(1) \times B\mathrm{Spin}(n)$ and $B(\mathrm{O}(1) \times \mathrm{Spin}(n))$ with Bj_n is homotopic to g_n because both maps, the composition and g_n, classify the universal vector bundle $\gamma_\infty \times \mathbb{R}^n_{\mathrm{Spin}}$. We denote the isometric classifying vector bundle map with G_n. These maps induce a map between the Thom-spectra, and we can summarise the previous discussions by the following theorem, which leads to the isomorphism described in [ABP69] and strengthens a result in [KT90, Lemma 6].

Theorem 1.4.10. *The map*

$$\mathrm{Th}(G)\colon \mathbf{MO}(1) \wedge \mathbf{MSpin} \to \mathbf{MPin}^+$$

is a homotopy equivalence.

In order to present the desired isomorphism, we have to determine the Thom space $\mathbf{MO}(1)$.

Lemma 1.4.11. *Let $\gamma_n \to \mathbb{R}P^n$ be the tautological line bundle. Then we have*

1. $\mathrm{Th}(\gamma_n)$ *is homeomorphic to* $\mathbb{R}P^{n+1}$.
2. $\mathbf{MO}(1) = \mathrm{Th}(\gamma_\infty)$ *is homeomorphic to* $\mathbb{R}P^\infty$

Proof. Observe that the tautological line bundle can be described by

$$\gamma_n = S^n \times_{\mathbb{Z}_2} \mathbb{R} = (S^n \times \mathbb{R})/\,(x,t) \sim (-x,-t).$$

Thus, the disc and the sphere bundle are given by

$$D(\gamma_n) = S^n \times_{\mathbb{Z}_2} [-1,1]$$
$$S(\gamma_n) = S^n \times_{\mathbb{Z}_2} \times\{-1,1\}$$

and we get an homeomorphism

$$D(\gamma_n)/\,S(\gamma_n) \to \mathbb{R}P^{n+1}$$

induced by

$$[x,t] \mapsto \left[\sqrt{1-t^2} : t \cdot x\right].$$

This shows the first part of the lemma. The second part is nothing but passing n to infinity. □

Note that we have under this isomorphism

$$\mathbf{MO}(1) = \mathbb{R}P^\infty \supsetneq \{[0:x] \mid x \in S^\infty\} = B\mathbb{Z}_2 = BO(1).$$

Corollary 1.4.12. *We have the following description of* Pin^- *bordism in terms of* Spin *bordism.*

$$\Omega_n^{\mathrm{Pin}^-}(X) \cong \Omega_{n+1}^{\mathrm{Spin}}(\mathbb{R}P^\infty \times X, X)$$

Proof. Let x_0 be a base point of $\mathbb{R}P^\infty$ and $X_+ = X \sqcup \{\infty\}$. Then we have

$$\begin{aligned}
(\mathbb{R}P^\infty \times X)_+ / X_+ &= (\mathbb{R}P^\infty \times X \sqcup \{\infty\}) / (x_0 \times X \sqcup \{\infty\}) \\
&= \mathbb{R}P^\infty \times X / \{x_0\} \times X \\
&= (\mathbb{R}P^\infty \times (\{\infty\} \sqcup X)) / (\mathbb{R}P^\infty \times \{\infty\} \cup \{x_0\} \times X) \\
&= (\mathbb{R}P^\infty \times X_+) / \mathbb{R}P^\infty \vee X_+ \\
&= \mathbb{R}P^\infty \wedge X_+.
\end{aligned}$$

From this equation we derive the isomorphism

$$\begin{aligned}
\Omega_n^{\mathrm{Pin}^-}(X) &= \lim_{l\to\infty} \pi_{n+l}(M\mathrm{Pin}_l^- \wedge X_+) \\
&\cong \lim_{l\to\infty} \pi_{n+l}(M\mathrm{Spin}_{l-1} \wedge \mathbb{R}P^\infty \wedge X_+) \\
&= \lim_{l\to\infty} \pi_{n+l}(M\mathrm{Spin}_{l-1} \wedge (\mathbb{R}P^\infty \times X)_+ / X_+) \\
&= \Omega_{n+1}^{\mathrm{Spin}}(\mathbb{R}P^\infty \times X, X).
\end{aligned}$$

□

The isomorphism given in Corollary 1.4.12 is rather abstract and has actually $\Omega_{n+1}^{\mathrm{Spin}}(\mathbb{R}P^\infty \times X, X)$ as domain. So, it is not useful for practical calculations. However, there is another homomorphism, which will turn out to be very useful in the determination of $[S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}]$ in $\Omega_2^{\mathrm{Pin}^-}(\mathrm{pt})$, see Corollary 1.5.11.

Definition 1.4.13. Let $\Phi\colon \det TM \to \gamma_\infty$ be a vector bundle map covering the classifying map of $\det TM$. Then $\mathrm{Th}(\Phi) \circ \exp^{-1}\colon \mathcal{S}(\mathrm{M}) \setminus \mathrm{im}(\sigma_{-1}) \to \mathrm{Th}(\gamma_\infty)$ has a continuous extension to $\mathcal{S}(\mathrm{M})$ by sending the image of σ_{-1} to infinity. Denote this extension with $\mathfrak{t}_{\det TM}$.

Note that different classifying maps give homotopic extensions.

Theorem 1.4.14. *The assignment*

$$\begin{aligned}
\mathfrak{S}\colon \Omega_n^{\mathrm{Pin}^-}(X) &\to \Omega_{n+1}^{\mathrm{Spin}}(\mathbb{R}P^\infty \times X) \\
[M, f] &\mapsto [\mathcal{S}(M), \mathfrak{t}_{\det TM} \times f \circ p]
\end{aligned}$$

is a group homomorphism.

Proof. Let (W, F) be a singular Pin^- bordism between (M_0, f_1) and (M_1, f_1). Then $(\mathcal{S}(W), F \circ p)$ is a singular boundary of $(\mathcal{S}(\mathrm{M})(M_0) \sqcup \mathcal{S}(M_1), f_0 \sqcup f_1)$. It is even a Pin^--boundary because the Pin^- structures $\partial\mathcal{S}(W)$ and $\mathcal{S}(\partial W)$

agree by the uniqueness Theorems 1.2.15 and 1.2.20. The construction of $\mathfrak{t}$ is fibrewise, so we have the identity

$$\mathfrak{t}_{\det TW|_{\partial\mathcal{S}(W)}} = \mathfrak{t}_{\det TM_0} \sqcup \mathfrak{t}_{\det TM_1}.$$

Furthermore, the homotopy class of $\mathfrak{t}_{\det TM}$ depends only on M. Thus, the given assignment is well-defined and, because of

$$(\mathcal{S}(M_0 \sqcup M_1), (f_0 \sqcup f_1) \circ p) = (\mathcal{S}(M_0) \sqcup \mathcal{S}(M_1), f_0 \circ p \sqcup f_1 \circ p),$$

it is even a homomorphism. □

Lemma 1.4.15. *If M is a* Spin *manifold, then* $(\mathcal{S}(M), \mathfrak{t}_{\det TM}) = (M \times S^1, \mathrm{pr}_2)$*, where we use the interpretation* $S^1 = \mathbb{R}P^1 \subseteq \mathbb{R}P^\infty$*.*

Proof. Since M is oriented, we find a section $s\colon M \to M^{or}$. This section induces a diffeomorphism

$$\begin{aligned} M \times S^1 &\to M^{or} \times_{\mathbb{Z}_2} S^1 = \mathcal{S}(M) \\ (m, z) &\mapsto [s(m), z]. \end{aligned}$$

Orientiability of TM also implies that we can choose the classifying map of $\det TM$ to be constant with value $[1 : 0 \dots] \in \mathbb{R}P^\infty$. Thus, the associated map on Thom spaces is induced by

$$\begin{aligned} D(\det TM) = M^{or} \times_{\mathbb{Z}_2} [-1, 1] &\to \mathbb{R}P^\infty = \mathrm{Th}(\gamma_\infty) \\ [\tilde{m}, t] &\mapsto \left[t : \sqrt{1-t^2} : 0 : \dots\right]. \end{aligned}$$

By pulling $\mathfrak{t}_{\det TM}$ back with this diffeomorphism, we obtain the continuous map

$$(m, \exp(\mathrm{i}\varphi)) \mapsto \left[\frac{\varphi}{\pi} : \sqrt{1 - \left(\frac{\varphi}{\pi}\right)^2} : 0 : \dots\right],$$

which can be interpreted as pr_2 under the identification $S^1 = \mathbb{R}P^1$. □

1.5. The Pin bordism coefficients

In this section, we are going to calculate the coefficient groups $\Omega_n^{\mathrm{Pin}^-}(\mathrm{pt})$ for $n \leq 4$. Using the isomorphism $\Omega_n^{\mathrm{Pin}^-}(\mathrm{pt}) \cong \Omega_{n+1}^{\mathrm{Spin}}(\mathbb{R}P^\infty, \mathrm{pt})$, this task is equivalent to determine the relative Spin bordism group of $\mathbb{R}P^\infty$ for $n \leq 5$, which can be done by applying the Atiyah-Hirzebruch spectral sequence to $\Omega_*^{\mathrm{Spin}^-}(\mathbb{R}P^\infty)$, and then splitting off the Spin coefficient groups. The notation we use for the spectral sequence calculations is introduced in Appendix C.

The next theorem summarises the results of the involved calculations.

Theorem 1.5.1. *The first six relative* Spin *bordism groups of* $\mathbb{R}P^\infty$ *are given by:*

$$\begin{aligned} \Omega_0^{\mathrm{Spin}}(\mathbb{R}P^\infty, \mathrm{pt}) &\cong 0, & \Omega_3^{\mathrm{Spin}}(\mathbb{R}P^\infty, \mathrm{pt}) &\cong \mathbb{Z}_8, \\ \Omega_1^{\mathrm{Spin}}(\mathbb{R}P^\infty, \mathrm{pt}) &\cong \mathbb{Z}_2, & \Omega_4^{\mathrm{Spin}}(\mathbb{R}P^\infty, \mathrm{pt}) &\cong 0, \\ \Omega_2^{\mathrm{Spin}}(\mathbb{R}P^\infty, \mathrm{pt}) &\cong \mathbb{Z}_2, & \Omega_5^{\mathrm{Spin}}(\mathbb{R}P^\infty, \mathrm{pt}) &\cong 0. \end{aligned}$$

Corollary 1.5.2. *For the* Pin$^-$ *bordism coefficients we get the following isomorphisms.*[3]

$n =$	0	1	2	3	4
$\Omega_n^{\mathrm{Pin}^-}(\mathrm{pt})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	0	0

The generator of $\Omega_1^{\mathrm{Pin}^-}(\mathrm{pt})$ *is* $[S^1_{\mathrm{Lie}}]$ *and the generator of* $\Omega_2^{\mathrm{Pin}^-}(\mathrm{pt})$ *is* $[\mathbb{R}P^2]$.

Proof. It remains to prove the statement about the generators. Every compact manifold of dimension one is diffeomorphic to S^1, which has two different Pin$^-$ structures. But S^1 equipped with the trivial Pin$^-$ structure is a boundary of D^2. So, the generator must be S^1_{Lie}. Theorem 1.3.9 states that $[\mathbb{R}P^2]$ is a generator of $\Omega_2^{\mathrm{Pin}^-}(\mathrm{pt})$. □

Before we start with the proof, let us recall from Appendix A that $\Omega_*^{\mathrm{Spin}}(\mathrm{pt})$ is a graded ring acting on $\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^\infty)$. This action turns the corresponding Atiyah-Hirzebruch spectral sequence into a spectral sequence of $\Omega_*^{\mathrm{Spin}}(\mathrm{pt})$-modules, which converges to $\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^\infty)$ as a $\Omega_*^{\mathrm{Spin}}(\mathrm{pt})$-module. Furthermore, $\Omega_1^{\mathrm{Spin}}(\mathrm{pt}) \cong \mathbb{Z}_2$ is generated by $[S^1_{\mathrm{Lie}}]$ and $\Omega_2^{\mathrm{Spin}}(\mathrm{pt}) \cong \mathbb{Z}_2$ is generated by $[S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}]$; we conclude that

$$\Omega_1^{\mathrm{Spin}}(\mathrm{pt}) \xrightarrow{\cdot \times [S^1_{\mathrm{Lie}}]} \Omega_2^{\mathrm{Spin}}(\mathrm{pt})$$

is an isomorphism.

From the coefficients listed in Example B.3.2, we are in the position to determine the groups $E^2_{p,q} = H_p(\mathbb{R}P^\infty, \Omega_q^{\mathrm{Spin}}(\mathrm{pt}))$, which are partially listed in Figure 1.1. The next step is to determine the differentials. A list of all

4	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
3	0	0	0	0	0	0
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
	0	1	2	3	4	5

Figure 1.1. The second page $E^2_{p,q} = H_p\left(\mathbb{R}P^\infty; \Omega_q^{\mathrm{Spin}}(\mathrm{pt})\right)$ for $p \leq 5$ and $q \leq 4$ of the Atiyah-Hirzebruch spectral sequence approximating $\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^\infty)$.

possibly non-trivial differentials of the second page we are interested in is provided in the next theorem.

Theorem 1.5.3. *Let* (p, q) *be a pair from the table below. Under the identification* $E^2_{p,q} \cong \mathbb{Z}_2 \cong E^2_{p-2,q+1}$, *the differentials corresponds to*

$(p,q) =$	$(2,1)$	$(3,0)$	$(3,1)$	$(4,1)$	$(5,0)$	$(5,1)$
$d^2_{p,q} =$	0	0	0	id	id	id

[3] For a complete list see [ABP69]

Before we prove this theorem, let us harvest its consequences. The differentials listed in Theorem 1.5.3 yield partial results for the third page listed in Figure 1.2. Since any differential mapping into the first column must be

4	$\mathbb{Z}$	$\mathbb{Z}_2$				
3	0	0	0			
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0		
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	
0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0
	0	1	2	3	4	5

Figure 1.2. The third page $E^3_{p,q}$ for $p \leq 5$ and $q \leq 4$ of the Atiyah-Hirzebruch spectral sequence approximating $\Omega^{\mathrm{Spin}}_*(\mathbb{R}P^\infty)$.

zero, in particular $d^3_{3,0}$, we observe that $d^r_{p,q} = 0$ for every $r \geq 3$ as long as $p+q \leq 5$. Therefore, we conclude $E^3_{p,q} \cong E^\infty_{p,q}$ for $p+q \leq 4$. Using a characteristic number argument, we will show that the generator of $E^3_{1,4}$ must not survive and hence, $E^\infty_{1,4} = 0$. Thus, we do not need to determine further differentials to write down the part of the infinity-page we are interested in. Since $E^\infty_{0,q} = F_{0,q} = \Omega^{\mathrm{Spin}}_q(\mathrm{pt}) \subseteq \Omega^{\mathrm{Spin}}_q(\mathbb{R}P^\infty)$ always splits off,

4	$\mathbb{Z}$	0				
3	0	0	0			
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0		
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	
0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0
	0	1	2	3	4	5

Figure 1.3. The infinity-page $E^\infty_{p,q}$ for $p \leq 5$ and $q \leq 4$ of the Atiyah-Hirzebruch spectral sequence approximating $\Omega^{\mathrm{Spin}}_*(\mathbb{R}P^\infty)$.

there is only one non-trivial extension problem to solve, namely, to determine $\Omega^{\mathrm{Spin}}_3(\mathbb{R}P^\infty) = \Omega^{\mathrm{Spin}}_3(\mathbb{R}P^\infty, \mathrm{pt})$ from the informations provided by the groups on the fourth diagonal $\{E^\infty_{p,q} \,|\, p+q=3\}$. But we know from Theorem 1.3.9 that $\Omega^{\mathrm{Spin}}_3(\mathbb{R}P^\infty, \mathrm{pt}) \cong \Omega^{\mathrm{Pin}^-}_2(\mathrm{pt})$ is cyclic. Therefore, $\Omega^{\mathrm{Pin}^-}_2(\mathrm{pt})$ is isomorphic to $\mathbb{Z}_8$. The generator is represented by the real projective plane rather than the Klein bottle. This results contradicts a statement in [Pet68] on page 34. Thus, we verified the equations in Theorem 1.5.1.

Now, we will fulfil our duty to fill the gaps, namely, to determine the differentials and to verify $E^\infty_{1,4} = 0$.

Lemma 1.5.4. $d^2_{2,1} = 0$.

Proof. Since $d^2_{2,1}$ maps to $E^2_{0,2}$, a group in the column indexed by $p = 0$, it must vanish by Corollary C.0.22. □

Using $\Omega^{\mathrm{Spin}}_*(\mathrm{pt})$-equivariance of the differentials, we deduce the following lemma.

Lemma 1.5.5.

$$d^2_{p,1} = 0 \Leftrightarrow d^2_{p,0} = 0, \text{ for every odd } p.$$
$$d^2_{p,1} = 0 \Leftrightarrow d^2_{p,2} = 0, \text{ for every } p.$$

Proof. Let either $q = 1$, or $q = 0$ and p odd. Then the multiplication with $[S^1_{\mathrm{Lie}}] \in \Omega^{\mathrm{Spin}}_1(\mathrm{pt})$ gives an isomorphism

$$E^2_{p,q} = H_p(\mathbb{R}P^\infty; \Omega^{\mathrm{Spin}}_q(\mathrm{pt})) \xrightarrow[\cong]{\cdot\times[S^1_{\mathrm{Lie}}]} H_p(\mathbb{R}P^\infty; \Omega^{\mathrm{Spin}}_{q+1}(\mathrm{pt})) = E^2_{p,q+1}.$$

The differential is $\Omega^{\mathrm{Spin}}_*(\mathrm{pt})$-equivariant, so we obtain a commutative square

$$\begin{array}{ccc} \mathbb{Z}_2 = E^2_{p,q} & \xrightarrow{d^2_{p,q}} & E^2_{p-2,q+1} = \mathbb{Z}_2 \\ \cong\downarrow \cdot\times[S^1_{\mathrm{Lie}}] & & \cong\downarrow \cdot\times[S^1_{\mathrm{Lie}}] \\ \mathbb{Z}_2 = E_{p,q+1} & \xrightarrow[d_{p,q+1}]{} & E^2_{p-2,q} \end{array}$$

with vertical isomorphisms; the statement follows. □

Lemma 1.5.6. $d^2_{3,0} = 0$ *and* $d^2_{3,1} = 0$.

Proof. Assume that $d^2_{3,0} \neq 0$. Then $E^3_{3,0} = \ker d^3_{3,0} = 0$ and, therefore, $E^\infty_{3,0} = 0$. Since $E^\infty_{3,0} = F_{3,0}/F_{2,1}$, the previous conclusion implies

$$\Omega^{\mathrm{Spin}}_3(\mathbb{R}P^\infty) = F_{2,1} = \mathrm{im}[\Omega^{\mathrm{Spin}}_3(\mathbb{R}P^2) \xrightarrow{\iota_*} \Omega^{\mathrm{Spin}}_3(\mathbb{R}P^\infty)].$$

In other words, every singular Spin manifold of dimension three is bordant to a singular Spin manifold with image in $\mathbb{R}P^2$. In particular, $(\mathbb{R}P^3, \iota)$ with the canonical inclusion $\iota\colon \mathbb{R}P^3 \hookrightarrow \mathbb{R}P^\infty$ would be bordant to a singular Spin manifold (M, f), where f takes values only values in $\mathbb{R}P^2$.

Applying this observation to the generalised Hurewicz map, we deduce

$$\iota_*[\mathbb{R}P^3] = \mathrm{incl}_* \circ f_*[M] = 0$$

because it factors through $H_3(\mathbb{R}P^2; \mathbb{Z}_2) = 0$. But this is a contradiction because $\iota_*\colon H_3(\mathbb{R}P^3) \to H_3(\mathbb{R}P^\infty)$ is an isomorphism.

Now, the equality $d^2_{3,1} = 0$ follows from equivariance, see Lemma 1.5.5. □

Corollary 1.5.7. $E^\infty_{1,2} = E^3_{1,2}$.

Proof. The differentials $d^r_{1,2}$ vanish for every $r \geq 2$ because their ranges lie in the column indexed by $q \leq 0$. Similarly, all $d^r_{1+r,2-r+1}$ vanishes for $r > 3$ vanishes because their domains are zero. Additionally, $d^3_{4,0} = 0$ because $E^3_{4,0} = E^2_{4,0} = 0$, and the claim follows. □

Lemma 1.5.8. $d^2_{4,1} \neq 0$.

Proof. Consider the homology long exact sequence

$$\cdots \xrightarrow{\Delta} \Omega_4^{\mathrm{Spin}}(\mathbb{R}P^2, \mathrm{pt}) \longrightarrow \Omega_4^{\mathrm{Spin}}(\mathbb{R}P^4, \mathrm{pt}) \longrightarrow \Omega_4^{\mathrm{Spin}}(\mathbb{R}P^4, \mathbb{R}P^2) \xrightarrow{\Delta} \cdots.$$

By applying the Atiyah-Hirzebruch spectral sequence to $\mathbb{R}P^2$, we observe $\Omega_4^{\mathrm{Spin}}(\mathbb{R}P^2, \mathrm{pt}) \cong \mathbb{Z}_2$. Indeed, only the first three columns of the second page are non-zero. Therefore, every differential $d^2_{p,q}$ having a non-zero target space and a non-zero domain must fulfil $p = 2$. But $d^2_{2,q}$ is always the zero map because its target space lies in the column indexed by $p = 0$, see Corollary C.0.22. The higher differentials $d^r_{p,q}$ with $r \geq 3$ always die because either their domain or their target is zero. Consequently, $E^2_{p,q} = E^\infty_{p,q}$. Since $E^\infty_{3,1} = E_{3,1} = 0$, we have $\Omega_4^{\mathrm{Spin}}(\mathbb{R}P^2, \mathrm{pt}) \cong E^2_{2,2} \cong \mathbb{Z}_2$.

Next, we consider the group $\Omega_4^{\mathrm{Spin}}(\mathbb{R}P^4, \mathbb{R}P^2) \cong \Omega_4^{\mathrm{Spin}}(\mathbb{R}P^4/\mathbb{R}P^2, \mathrm{pt})$. Since $\mathbb{R}P^4/\mathbb{R}P^2$ is homotopy equivalent to $S^3 \cup_\varphi D^4$, where $\varphi\colon S^3 \to S^3$ is a map of degree two, the corresponding Atiyah-Hirzebruch spectral sequence for this space has the second page partially presented in Figure 1.4. Any differential with target in the column indexed by $p = 0$ vanishes, so

4	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	0
3	0	0	0	0	0	0
2	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
1	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
0	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	0
	0	1	2	3	4	5

Figure 1.4. The second page $E^2_{p,q}$ of the Atiyah-Hirzebruch spectral sequence for approximating $\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^4/\mathbb{R}P^2)$.

we conclude that the second page is also the infinity-page. We read off that $\Omega_4^{\mathrm{Spin}}(\mathbb{R}P^4, \mathbb{R}P^2) \cong \Omega_4^{\mathrm{Spin}}(\mathbb{R}P^4/\mathbb{R}P^2, \mathrm{pt}) \cong \mathbb{Z}_2$. Thus, $\Omega_4^{\mathrm{Spin}}(\mathbb{R}P^4, \mathrm{pt}) \to \Omega_4^{\mathrm{Spin}}(\mathbb{R}P^4, \mathbb{R}P^2)$ must either be zero or be surjective. From exactness we deduce injectivity for the morphism $\Omega_3^{\mathrm{Spin}}(\mathbb{R}P^2, \mathrm{pt}) \xrightarrow{i_*} \Omega_3^{\mathrm{Spin}}(\mathbb{R}P^4, \mathrm{pt})$, and, therefore, injectivity for

$$\Omega_3^{\mathrm{Spin}}(\mathbb{R}P^3) \supseteq F_{1,2} \xrightarrow{i_{1,2}} F_{1,2} \subseteq \Omega_3^{\mathrm{Spin}}(\mathbb{R}P^4).$$

But this is a contradiction because

$$F_{1,2} = E^\infty_{1,2} = E^3_{1,2} \to 0 = E^3_{1,2} = F_{1,2} \subseteq \Omega_3^{\mathrm{Spin}}(\mathbb{R}P^4).$$

To see that the right $E^3_{1,2}$ is zero, we use that $\mathrm{incl}\colon \mathbb{R}P^4 \hookrightarrow \mathbb{R}P^\infty$ induces a natural isomorphism on the second pages of the corresponding AHSS in degree $p \leq 3$. So, by Lemma 1.5.6 we have $d^2_{3,1} \neq 0$ for the AHSS approximating $\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^4)$.

4	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0
3	0	0	0	0	0	0
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0
	0	1	2	3	4	5

Figure 1.5. The second page $E^2_{p,q}$ of the Atiyah-Hirzebruch spectral sequence for approximating $\Omega^{\mathrm{Spin}}_*(\mathbb{R}P^4)$.

Consequently, $\Omega^{\mathrm{Spin}}_4(\mathbb{R}P^4, \mathrm{pt}) \to \Omega^{\mathrm{Spin}}_4(\mathbb{R}P^4, \mathbb{R}P^2)$ is the zero map. Exactness then implies that $\mathbb{Z}_2 \cong \Omega^{\mathrm{Spin}}_4(\mathbb{R}P^2, \mathrm{pt})$ surjects onto $\Omega^{\mathrm{Spin}}_4(\mathbb{R}P^4, \mathrm{pt})$.

Now consider the second page of the AHSS for $\Omega^{\mathrm{Spin}}_*(\mathbb{R}P^4)$. By the previous lemmas, we already know that $d^r_{3,0} = 0$ for every $r \geq 0$, so $E^\infty_{3,1} = E^2_{3,1} \cong \mathbb{Z}_2$. Since $E^\infty_{2,2} = E^2_{2,2}/\operatorname{im} d^2_{4,1}$, we conclude that $\operatorname{im} d^2_{4,1} \neq \{0\}$; otherwise the group $\Omega^{\mathrm{Spin}}_4(\mathbb{R}P^4, \mathrm{pt})$ would consists of four elements and that is a contradiction.

Since the inclusion $\mathbb{R}P^4 \hookrightarrow \mathbb{R}P^\infty$ induces natural isomorphisms $E^2_{p,q} \to E^2_{p,q}$ between the corresponding spectral sequences if $p \leq 4$, the differential $d^2_{4,1}$ must also be non-zero in the AHSS for $\Omega^{\mathrm{Spin}}_*(\mathbb{R}P^\infty)$. □

Lemma 1.5.9. *$d^2_{5,0} \neq 0$ and $d^2_{5,1} = 0$.*

Proof. Since $d^2_{4,1} \neq 0$, the differential $d^3_{5,0} \colon E^3_{5,0} \to E^3_{2,2} = \{0\}$ vanishes necessarily. This implies $E^\infty_{5,0} = E_{5,0}/\operatorname{im} d^2_{5,0}$. So, if we assume that $d^2_{5,0} = 0$, then the isomorphism induced by forgetting the Spin structure

$$^{\mathrm{Spin}}E^2_{5,0} = H_5(\mathbb{R}P^\infty, \Omega^{\mathrm{Spin}}_0) \to H_5(\mathbb{R}P^\infty, \Omega^{\mathrm{SO}}_0) = {}^{\mathrm{SO}}E^2_{5,0}$$

would give rise to an isomorphism

$$^{\mathrm{Spin}}E^\infty_{5,0} \to {}^{\mathrm{SO}}E^\infty_{5,0},$$

and the map $^{\mathrm{Spin}}F_{5,0} \to {}^{\mathrm{SO}}F_{5,0}$ would not factor through $^{\mathrm{SO}}F_{4,1} = {}^{\mathrm{SO}}F_{3,2}$. Consequently, there would be an element of the form

$$[\mathbb{R}P^5, \mathrm{incl}] + \sum_{n=0}^{2} [M^n, f_n],$$

which must be hit by a singular Spin manifold. Here, (M^n, f) denotes an oriented singular manifold of dimension n. But this can be ruled out using characteristic numbers. Indeed, under the identification $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$ and $H^*(\mathbb{R}P^5; \mathbb{Z}_2) \cong \mathbb{Z}_2[a]/\langle a^6 \rangle$ we have

$$w_2(\mathbb{R}P^2) \cup \mathrm{incl}^* x^3 + \sum_{n=0}^{2} f_n^*(x^3) \cup w_2(M^n) = a^2 \cup a^3 = a^5$$

because $f_n^*(x^3) \in H^3(M^n;\mathbb{Z}_2) = 0$. By Poincaré duality, we conclude that the characteristic number associated to this class is 1 instead of 0. So, this class cannot be hit by Spin manifold and we derived a contradiction.

Consequently, $d^2_{5,0}$ is indeed non-zero and we have finally derived the first part of the statement. The second part $d^2_{5,1}$ follows from equivariance. □

Lemma 1.5.10. $E^\infty_{1,4} = 0$.

Proof. Assume the contrary, then $E^\infty_{1,4} = E^2_{1,4} \cong \mathbb{Z}_2$. A comparison with oriented bordism will lead to a contradiction. Forgetting the Spin structure gives an isomorphism

$$^{\mathrm{Spin}}E_{1,4} \to {}^{\mathrm{SO}}E^2_{1,4} = {}^{\mathrm{SO}}E^\infty_{1,4}.$$

The groups on the right-hand-side are generated by $[\mathbb{C}P^2 \times \mathbb{R}P^1, \mathrm{pr}_2]$. This can be deduced from the $\Omega^{\mathrm{SO}}_*(\mathrm{pt})$-action on the Atiyah-Hirzebruch spectral sequence. But from this isomorphism we conclude that there is an element in ${}^{\mathrm{SO}}F_{1,4}$ that has the form

$$[\mathbb{C}P^2 \times \mathbb{R}P^1, \mathrm{pr}_2] + [M^5, \mathrm{const}],$$

and must be hit by a singular Spin manifold. Again, this can be ruled out by an characteristic number argument. Identify $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2)$ with $\mathbb{Z}_2[x]$ and $H^*(\mathbb{C}P^2 \times \mathbb{R}P^1;\mathbb{Z}_2)$ with $\mathbb{Z}_2[a,b]/\langle a^3, b^2\rangle$. We find

$$\begin{aligned} &\mathrm{pr}_2^*(x) \cup w_2^2(\mathbb{C}P^2 \times \mathbb{R}P^1) + \mathrm{const}^*(x) \cup w_2^2(M^5) \\ &= \mathrm{pr}_2^*(x) \cup \big(w_2(\mathbb{C}P^2) \times 1 + w_1(\mathbb{C}P^2) \times w_1(\mathbb{R}P^1) + 1 \times w_2(\mathbb{R}P^1)\big)^2 \\ &= (1 \otimes b) \cdot (a^2 \otimes 1) = a^2 \otimes b \neq 0 \end{aligned}$$

and conclude from Poincaré duality that the induced generalised Stiefel Whitney number is not zero. Therefore, this element cannot be represented by a singular Spin manifold, and we have derived the desired contradiction. □

As an application of the previous calculations, we will verify that the torus with the 'bad' Spin structure is also a non trivial element in $\Omega_2^{\mathrm{Pin}^-}(\mathrm{pt})$.

Corollary 1.5.11. $[S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}] \neq 0 \in \Omega_2^{\mathrm{Pin}^-}(\mathrm{pt})$.

Proof. It suffices to find a homomorphism that maps $[S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}]$ not to $0 \in \Omega_3^{\mathrm{Spin}}(\mathbb{R}P^\infty, \mathrm{pt}) = \Omega_3^{\mathrm{Spin}}(\mathbb{R}P^\infty)$. Luckily, the Spinification does the job. Indeed, since $S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}$ is a Spin manifold, the homomorphism $\mathfrak{S}$ maps $[S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}]$ to $[S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}} \times S^1, \mathrm{pr}_3]$, which is not zero because it generates the group $E^\infty_{1,2} = E^2_{1,2}$ in the Atiyah-Hirzebruch spectral sequence associated to $\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^\infty)$. Indeed, $[\mathbb{R}P^1, \mathrm{incl}] = [S^1, \mathrm{incl}]$, endowed with any Spin structure, generates

$$\Omega_1^{\mathrm{Spin}}(\mathbb{R}P^\infty, \mathrm{pt}) = F_{1,0}/F_{0,1} = E^\infty_{1,0} = E^2_{1,0} \neq 0.$$

So, by the discussion above, multiplying with $[S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}]$ gives an isomorphism between $E^2_{1,0}$ and $E^2_{1,2} = E^\infty_{1,2}$. Thus, $[S^1 \times S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}, \mathrm{pr}_1]$ is also a generator of $E^\infty_{1,2} \neq 0$ because changing the order gives a Spin preserving diffeomorphism between the two representing singular Spin manifolds. □

Chapter 2

The classifying space BO(2)

The aim of this chapter is to give a detailed discussion of the geometrical and topological structure of the classifying spaces $B\mathbb{Z}_2$, $B\mathrm{SO}(2)$, and $B\mathrm{O}(2)$. It is commonly known that $B\mathbb{Z}_2$ and $B\mathrm{SO}(2) = BS^1$ have $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$ as geometric models. In [Mil74, p.65] it is shown that $Gr(\infty, n)$ provides a model for $B\mathrm{O}(n)$; however, the suitable choice for our model will be $\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$. The advantages of this twisted product is that its cellular structure is closer to the 'standard' cellular structures of $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$, which is why it simplifies calculations on the level of chain complexes and (co-)homology. Even better, the cellular structure we chose for this product consists of closed cells that are compact manifolds. This observation turns out to be fruitful for the calculation of the bordism groups presented in chapter 3.

The structure of this chapter can be summarised as follows. Firstly, we will show that $\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$ serves indeed as a model for $B\mathrm{O}(2)$ and that it is a $\mathbb{C}P^\infty$-fibre bundle over $\mathbb{R}P^\infty$. If we endow this model with the CW-structure we have described above, then all closed closed cells are the closed manifolds $\mathbb{C}P^m \times_{\mathbb{Z}_2} S^n$, which are known as Dold-manifolds and generators of the ring $\mathfrak{N}_* = \Omega_*^{\mathrm{O}}(\mathrm{pt})$ [Dol56]. Using this decomposition, we are able to calculate the differentials of the cellular (co-)complexes and, consequently, to determine the (co-)homology groups in any coefficients. Explicit results will be stated for the rings $\Lambda \in \{\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_8\}$.

In the last part of this chapter we take a closer look at the cells $P(m,n)$ itself. We will show that $P(m,n)$ generates a homology class dual to $w_1^n w_2^m$ and calculate its total Stiefel-Whitney class. This will tell us which of those manifolds are Pin, orientable, e.t.c.

After developing the content of the first four sections of this chapter, the author has noticed the existence of the very worth reading paper of Dold [Dol56], which contains many of the independently developed results here.

2.1. A twisted product as model

Before we give the precise definition of $\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$, let us recall that $\mathrm{O}(n) = \mathrm{SO}(n) \times \mathbb{Z}_2$ as sets, but the group structure is given by a semi-direct product rather than a direct product. Indeed, there is a short exact sequence

$$1 \longrightarrow \mathrm{SO}(n) \xrightarrow{\iota} \mathrm{O}(n) \underset{r}{\overset{\det}{\rightleftarrows}} \mathbb{Z}_2 \longrightarrow 1,$$

where $r(\pm 1) = \mathrm{diag}(\pm 1, 1, \ldots, 1) \in \mathrm{O}(n)$. Therefore, $\mathrm{O}(n) \cong \mathrm{SO}(n) \rtimes_\theta \mathbb{Z}_2$, where $\theta\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{SO}(n))$ is defined by $\theta(h)(n) = \iota^{-1}(r(h)\iota(n)r(h)^{-1})$. The group structure on $\mathrm{SO}(n) \rtimes_\theta \mathbb{Z}_2$ is then given by

$$(n_1, h_1).(n_2, h_2) = (n_1\theta(h_1)(n_2), h_1h_2)$$

and the isomorphism by $(n, h) \mapsto n \cdot r(h)$, as one can easily check.

If $n = 2$, then $\mathrm{O}(2) \cong S^1 \rtimes \mathbb{Z}_2 \curvearrowright \mathbb{C}$, where S^1 acts by complex multiplication and $(1, -1)$ as complex conjugation. It is not hard to check that the isomorphism carries this representation to the standard representation of $\mathrm{O}(2)$ on $\mathbb{R}^2$.

Definition 2.1.1. Let $\mathbb{Z}_2$ act on $\mathbb{C}P^\infty \times S^\infty$ via $(-1).([z], x) = ([\bar{z}], -x)$ and define

$$\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty = (\mathbb{C}P^\infty \times S^\infty)/\,\mathbb{Z}_2.$$

The concept of a semi direct product allows us to identify $\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$ as a model for $B\mathrm{O}(2)$ in a very transparent manner. If we interpret $S^\infty = S^{2\infty-1} \subseteq \mathbb{C}^\infty$, then S^1 acts on S^∞ by complex multiplication, and we obtain a S^1 fibre bundle over $S^\infty/S^1 = \mathbb{C}P^\infty$. By replacing S^1 with $\mathbb{Z}_2$, we identify S^∞ as a $\mathbb{Z}_2$-fibre bundle over $\mathbb{R}P^\infty$. Since S^∞ is contractible, these bundles must be universal (c.f. [tD08, Theorem 14.4.12]), and $\mathbb{C}P^\infty$ and $\mathbb{R}P^\infty$ are classifying spaces for the groups S^1 and $\mathbb{Z}_2$, respectively. An analogous result for $B\mathrm{O}(2)$ is given in the next lemma.

Lemma 2.1.2. *The composition of the canonical projections*

$$S^\infty \times S^\infty \xrightarrow{\pi\times\mathrm{id}} \mathbb{C}P^\infty \times S^\infty \xrightarrow{\kappa} \mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$$

induces the structure of a principal $\mathrm{O}(2)$*-bundle on* $S^\infty \times S^\infty$*. Consequently,* $\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$ *serves as a model for* $B\mathrm{O}(2)$*.*

Proof. A right action $\mathrm{O}(2) = S^1 \rtimes \mathbb{Z}_2$ is given by

$$\begin{aligned}(S^\infty \times S^\infty) \times (S^1 \rtimes \mathbb{Z}_2) &\to S^\infty \times S^\infty,\\ ((z, x), (w, y)) &\mapsto (\theta(y)(z \cdot w), x \cdot y)\end{aligned}$$

as the following calculation shows:

$$\begin{aligned}(z, x).\left((w_1, y_1)(w_2, y_2)\right) &= (z, x).(w_1\theta(y_1)(w_2), y_1y_2)\\ &= \left(\theta(y_1y_2)\left(z \cdot w_1\theta(y_1)(w_2)\right), x \cdot y_1y_2\right)\\ &= \theta(y_2)\theta(y_1)\left((z \cdot w_1\theta(y_1)(w_2)\right), x \cdot y_1y_2)\\ &= \left(\theta(y_2)\left(\theta(y_1)(z \cdot w_1) \cdot w_2\right), x \cdot y_1y_2\right)\\ &= \left((z, x).(y_1, w_1)\right).(w_2, y_2).\end{aligned}$$

To see that $S^\infty \times S^\infty$ is locally trivial, consider the open subsets

$$U_m = \{[z] \mid z_m \neq 0\} \subseteq \mathbb{C}P^\infty \text{ and } V_n^\pm = \{x \mid \pm x_n > 0\} \subseteq S^\infty.$$

The bundle $S^\infty \times S^\infty$ will be trivial over the open subsets $[U_m \times V_n^\pm] \subseteq \mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$. Indeed, there are equivariant homeomorphisms given by

$$\begin{aligned} \Phi^{-1}_{m,n,\pm} \colon [U_m \times V_n^\pm] \times \mathrm{O}(2) &\to S^\infty \times S^\infty|_{[U_m \times V_n^\pm]} = U_m \times (V_n^+ \cup V_n^-), \\ ([u,v]; (z,x)) &\mapsto \theta(x)(\tilde{u} \cdot z, \tilde{v} \cdot x), \end{aligned}$$

where $(\tilde{u}, \tilde{v})$ is the unique representative of $[u,v]$ satisfying $\tilde{u}_m > 0$ and $\pm v_n > 0$. The previous calculation also shows the equivariance of these maps.

The diagram

$$\begin{array}{ccc} [U_m \times V_n^\pm] \times \mathrm{O}(2) & \longrightarrow & U_m \times (V_n^+ \cup V_n^-) \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \kappa \circ (\pi \times \mathrm{id})} & \\ & [U_m \times V_n^\pm] & \end{array}$$

commutes by construction, and one easily sees that $\Phi^{-1}|_{m,n,\pm}$ is a homeomorphism; thus, $S^\infty \times S^\infty$ is indeed a principal $\mathrm{O}(2)$-bundle over $\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$. Since the total space is contractible, $S^\infty \times S^\infty \to \mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$ serves indeed as a model for $E\mathrm{O}(2) \to B\mathrm{O}(2)$. □

Theorem 2.1.3. *For any $0 \leq m, n \leq \infty$, the projection map*

$$\begin{aligned} \pi \colon \mathbb{C}P^m \times_{\mathbb{Z}_2} S^n &\to \mathbb{R}P^n, \\ [[z], x] &\mapsto [x] \end{aligned}$$

turns $P(m,n)$ into a $\mathbb{C}P^m$-fibre bundle.

Proof. Similarly to the notation used in the last proof, we define $V_k := \{[x] \in \mathbb{R}P^n \mid x_k \neq 0\}$. Then the map

$$\begin{aligned} \Phi_k^{-1} \colon V_k \times \mathbb{C}P^k &\to [\mathbb{C}P^m \times V_k^+], \\ ([x],[z]) &\mapsto [z, \hat{x}] \end{aligned}$$

is a homeomorphism. Here, $\hat{x}$ is the representative of $[x]$ satisfying $x_k > 0$. Of course the relation $p \circ \Phi_k^{-1} = \mathrm{pr}_1$ is also satisfied. Consequently, π is a fibre bundle map with fibre $\mathbb{C}P^m$. □

There is a canonical section of π

$$\begin{aligned} \sigma_0 \colon \mathbb{R}P^n &\to P(m,n), \\ [x] &\mapsto [[1:0:\ldots], x], \end{aligned}$$

which is a cellular map. We will use this section in chapter 3 to produce a split in the spectral sequences for $B\mathrm{O}(2)$. However, it is convenient to know that any two sections of π are homotopic and, therefore, that the split will not depend on a specific choice.

Theorem 2.1.4. *Any two sections of $\pi \colon B\mathrm{O}(2) \to \mathbb{R}P^\infty$ are homotopic.*

Proof. Let σ be an arbitrary section of π. We will show that $\sigma^*(S^\infty \times S^\infty)$ is isomorphic to

$$\begin{aligned} \sigma_0^*(S^\infty \times S^\infty) &= \{([x], z, x) \mid \sigma_0([x]) = [[z], x]\} \\ &\underset{\mathrm{pr}_2}{\cong} \{(z_1, x) \mid x \in [x]\} = S^1 \times S^\infty. \end{aligned}$$

From the universal property of $B\mathrm{O}(2)$ it will follow that σ and σ_0 are homotopic.

First observe that we can find a continuous map $s = s_\sigma \colon S^\infty \to \mathbb{C}P^\infty$ satisfying $s(-x) = \overline{s(x)}$ and $\sigma([x]) = [s(x), x]$. Indeed, covering theory provides a lift

$$\begin{array}{ccccc} S^\infty & \xrightarrow{2:1} & \mathbb{R}P^\infty & \xrightarrow{\sigma} & B\mathrm{O}(2) \\ & \searrow & & & \uparrow{\scriptstyle 2:1} \\ & & & & \mathbb{C}P^\infty \times S^\infty, \end{array}$$

and projecting this lift onto the first component gives the desired function $S = s_\sigma$. Since S^∞ is contractible, our function s is homotopic to the constant map $S^\infty \to \{[1:0:\ldots]\} \subseteq \mathbb{C}P^\infty$ via a homotopy denoted by h.

Since $S^1 \to S^\infty \to \mathbb{C}P^\infty$ is a Serre fibration, we can lift this homotopy to S^∞ such that $H(\cdot, 1)$ maps everything to $e_1 = (1, 0, \ldots)$. Set $\tilde{s} = H(\cdot, 0)$. This is a lift of s by construction. Since $[[\tilde{s}(x)], x] = [[\tilde{s}(-x)], -x]$, we conclude

$$\tilde{s}(-x) = \rho(x) \cdot \overline{\tilde{s}(x)}$$

with $\rho(x) \in S^1$. Note that the map $x \mapsto \rho(x) = \langle \tilde{s}(x), \tilde{s}(-x) \rangle$ is a continuous map $S^\infty \to S^1$. Here, $\langle \cdot, \cdot \rangle$ refers to the standard bilinear form on $\mathbb{C}^\infty$ and not the hermitian form. From

$$\begin{aligned} \tilde{s}(x) = \tilde{s}(-(-x)) &= \rho(-x) \cdot \overline{\tilde{s}(-x)} \\ &= \rho(-x) \cdot \overline{\rho(x) \cdot \overline{\tilde{s}(x)}} \\ &= \rho(-x) \cdot \overline{\rho(x)} \cdot \tilde{s}(x), \end{aligned}$$

we derive $1 = \rho(-x)\overline{\rho(x)}$ or, equivalently,

$$\rho(x) = \rho(-x).$$

Since ρ is homotopic to the constant map, covering theory yields the existence of continuous square root $\rho^{1/2}$. Then

$$S_\sigma(x) := \left(\rho^{1/2}\right)^{-1} \cdot \tilde{s}(x)$$

satisfies $S_\sigma(-x) = \overline{S_\sigma(x)}$ and, we have therefore

$$\sigma([x]) = [[S_\sigma(x)], x].$$

With the help of S_σ we have a nice description of $\sigma^*(S^\infty \times S^\infty)$, namely

$$\begin{aligned} \sigma^*(S^\infty \times S^\infty) &= \{([x], z, v) \mid \sigma([x]) = [[z], v]\} \\ &= \{([x], \zeta \cdot S_\sigma(x), x) \mid \zeta \in S^1\} \\ &\underset{\mathrm{pr}_1}{\cong} \{(\zeta \cdot S_\sigma(x), x) \mid \zeta \in S^1\}. \end{aligned}$$

Now, an equivariant map between $\sigma^*(S^\infty \times S^\infty)$ and $S^1 \times S^\infty$ is given by

$$\begin{aligned} \sigma_0^*(S^\infty \times S^\infty) = S^1 \times S^\infty &\to \sigma^*(S^\infty \times S^\infty) \\ (\zeta, x) &\mapsto (\zeta \cdot S_\sigma(x), x). \end{aligned}$$

This map covers the identity on $\mathbb{R}P^\infty$ and is therefore a bundle equivalence. By the universal property of $BO(2)$, the maps σ and σ_0 must be homotopic. $\square$

2.2. The CW-structure and the (co-)homology of the model

Having introduced our geometric model for $BO(2)$, we take a closer look at its CW-structure. We will derive a cell decomposition for $BO(2)$ close to the standard cellular structures of $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$. Then we will use this decomposition to calculate the singular homology and cohomology groups of $BO(2)$ with coefficients in $\Lambda \in \{\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_8\}$.

Let us recall that $\mathbb{C}P^\infty$ has the skeleton $(\mathbb{C}P^\infty)^{(2k-1)} = (\mathbb{C}P^\infty)^{(2k)} = \mathbb{C}P^k$, and that the attaching map to glue a $2(k+1)$-disc to the $(2k+1)$-skeleton is given by the canonical projection $S^{2k+1} \to \mathbb{C}P^k = S^{2k+1}/S^1$. Since we have only one closed cell of dimension $2k$, we denote it with $\mathbb{C}P^k$, too.

For $\mathbb{R}P^\infty$, we have an analogous result, namely $(\mathbb{R}P^\infty)^{(k)} = \mathbb{R}P^k$, and the attaching map is given by the canonical projection $S^k \to \mathbb{R}P^k = S^k/\mathbb{Z}_2$.

To construct our preferred cell structure for $BO(2) = \mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$, recall that S^∞ has the cell decomposition into upper and lower hemispheres. The skeleton is given by $(S^\infty)^{(k)} = S^k$, and we attach two $(k+1)$-cells to the skeleton via the following pushout

$$\begin{array}{ccc} S^k \sqcup S^k & \xrightarrow{\mathrm{id}\sqcup\mathrm{id}} & S^k \\ \downarrow & & \downarrow \\ D^{k+1} \sqcup D^{k+1} & \xrightarrow[\Phi_0 \sqcup \Phi_1]{} & S^{k+1}. \end{array}$$

The characteristic maps are defined by

$$\Phi_j(x) = \left(x, (-1)^j \sqrt{1 - ||x||^2}\right).$$

We denote the closed k-cells with $e_j^k = \Phi_j(D^k)$.

The cell decomposition of $\mathbb{C}P^\infty \times S^\infty$ is given by the product cells $\mathbb{C}P^k \times e_j^{n-2k}$. Since $\mathbb{C}P^\infty$ and S^∞ have only countably many cells, the limit topology on $\mathbb{C}P^\infty \times S^\infty$ agrees with the product topology [SZ94, Chapter 4].

We obtain a CW-structure on $\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$ by applying the canonical projection $\kappa\colon \mathbb{C}P^\infty \times S^\infty \to \mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$. More precisely, $\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$ can be decomposed into the closed $(2m+n)$ cells $\kappa(\mathbb{C}P^m \times S^n) = P(m,n)$, which are known to be the Dold-manifolds. Indeed, using this cell decomposition we get for the skeleton $BO(2)^{(k)} = \bigcup P(m, k-2m)$. The characteristic map for $P(m,n)$ is given by

$$D^{2m+n} \xrightarrow{\approx} D^{2m} \times D^n \longrightarrow \mathbb{C}P^m \times e_0^n \xrightarrow{\kappa} P(m,n).$$

Even better, $P(m, n)$ is not only a closed cell in $BO(2)$ but also a subcomplex consisting of all closed cells $P(m', n')$ with $m' \leq m$ and $n' \leq n$. Furthermore, one can easily observe that the maps given in Theorem 2.1.4 are not just cellular maps but maps sending closed cells to closed cells. The covering map κ does the same, by construction.

The cohomology of $\mathrm{BO}(2)$**:**

Let us first recall the cellular chain complexes of $\mathbb{C}P^\infty$ and S^∞, and use these to calculate the cellular chain complex of $\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$.

Since $\mathbb{C}P^\infty$ has only cells of even dimension, we have

$$\mathcal{C}_q^{cell}(\mathbb{C}P^\infty; \Lambda) \cong \begin{cases} \Lambda, & \text{if } q \text{ is even,} \\ 0, & \text{if } q \text{ is odd,} \end{cases}$$

and the cellular differential is therefore always zero for algebraic reasons.

Using the hemisphere decomposition of S^∞, we get for the cellular groups $\mathcal{C}_q^{cell}(S^\infty; \Lambda) \cong \Lambda^2$, where the generators are given by the upper and lower hemisphere. With an appropriate orientation, the cellular differential is given by

$$\partial_q(e_j^q) = (-1)^j(e_0^{q-1} + e_1^{q-1}).$$

Since $\mathbb{C}P^\infty \times S^\infty$ consists of product cells we derive derive

$$\begin{aligned} \partial_q(\mathbb{C}P^k \times e_j^{q-2k}) &= \partial_{2k}\mathbb{C}P^k \times e_j^{q-2k} + (-1)^{2k} \cdot \mathbb{C}P^k \times \partial_{q-2k} e_j^{q-2k} \\ &= 0 + (-1)^j \cdot \mathbb{C}P^k \times (e_0^{q-2k-1} + e_1^{q-2k-1}) \\ &= (-1)^j \cdot \mathbb{C}P^k \times (e_0^{q-2k-1} + e_1^{q-2k-1}). \end{aligned}$$

We need an auxiliary lemma to calculate the cellular differential of the cellular complex of $\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$.

Lemma 2.2.1. *The complex conjugation* $\bar{\cdot}\colon \mathbb{C}P^k \to \mathbb{C}P^k$ *reverses the orientation if and only if* k *is odd.*

Proof. It is well known that a linear isometry $A \in \mathrm{O}(n+1)$ restricts to a continuous selfmap on S^n with mapping degree $\det(A)$.

Using this fact, we derive the statement from the following commutative diagram

$$\begin{array}{ccc} H_{2k}(\mathbb{C}P^k, \mathbb{C}P^{k-1}) & \xrightarrow{\bar{\cdot}_*} & H_{2k}(\mathbb{C}P^k, \mathbb{C}P^{k-1}) \\ \Phi_* \uparrow \cong & & \cong \uparrow \Phi_* \\ H_{2k}(D^{2k}, S^{2k-1}) & \xrightarrow{\bar{\cdot}_*} & H_{2k}(D^{2k}, S^{2k-1}) \\ \partial \downarrow \cong & & \cong \downarrow \partial \\ H_{2k-1}(S^{2k-1}) & \xrightarrow[= (-1)^k \cdot \mathrm{id}]{\bar{\cdot}_*} & H_{2k-1}(S^{2k-1}). \end{array}$$

□

Theorem 2.2.2. *The cellular complex of* $\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty$ *has the following description: For the groups we have*

$$\mathcal{C}_q^{cell}(\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty) \cong \Lambda^{\lfloor \frac{q}{2} + 1 \rfloor}$$

with basis $\{P(k, q-2k) \,|\, 0 \leq k \leq \lfloor \frac{q}{2} \rfloor\}$. *The differential is given by*

$$\partial_q^{cell}(P(k, q-2k)) = (1 + (-1)^{q-k}) \cdot P(k, q-2k-1)$$

Proof. The statement about the cellular groups follows immediately from the cell decomposition we gave above. It remains to verify the identity for the cellular differential.

Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{C}_q^{cell}(\mathbb{C}P^\infty \times S^\infty) & \xrightarrow{\partial_q} & \mathcal{C}_{q-1}^{cell}(\mathbb{C}P^\infty \times S^\infty) \\ \downarrow{\scriptstyle \kappa_q} & & \downarrow{\scriptstyle \kappa_{q-1}} \\ \mathcal{C}_q^{cell}(\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty) & \xrightarrow[\partial_q]{} & \mathcal{C}_{q-1}^{cell}(\mathbb{C}P^\infty \times_{\mathbb{Z}_2} S^\infty), \end{array}$$

and recall that we have defined $P(k, q-2k) = \kappa_q(\mathbb{C}P^k \times e_0^{q-2k})$. Note that $\kappa_q(\mathbb{C}P^k \times e_1^{q-2k})$ and $P(k, q-2k)$ may differ by a sign as the following computation shows:

$$\begin{aligned} \kappa_q(\mathbb{C}P^k \times e_0^{q-2k}) &= \kappa_q \circ (\bar{\cdot} \times -\mathrm{id})_q(\mathbb{C}P^k \times e_1^{q-2k}) \\ &= \kappa_q \circ (\bar{\cdot} \times \mathrm{id})_q \circ (\mathrm{id} \times -\mathrm{id})_q(\mathbb{C}P^k \times e_1^{q-2k}) \\ &= \kappa_q \circ (\bar{\cdot} \times \mathrm{id})_q \left((-1)^{q-2k-1}\mathbb{C}P^k \times e_0^{q-2k}\right) \\ &= (-1)^{q-2k-1}\kappa_q(\overline{\mathbb{C}P}^k \times e_0^{q-2k}) \\ &= (-1)^{q-2k-1}(-1)^k\kappa_q(\mathbb{C}P^k \times e_0^{q-2k}) \\ &= (-1)^{q-k}P(k, q-2k). \end{aligned}$$

Putting these informations together, we get

$$\begin{aligned} \partial_q(P(k, q-2k)) = \partial_q \circ \kappa_q(\mathbb{C}P^k \times e_0^{q-2k}) &= \kappa_{q-1}\left(\partial_q(\mathbb{C}P^k \times e_0^{q-2k})\right) \\ &= \kappa_{q-1}(\mathbb{C}P^k \times e_0^{q-1-2k} + \mathbb{C}P^k \times e_1^{q-1-2k}) \\ &= P(k, q-2k-1) + (-1)^{(q-1)-k-1}P(k, q-1-2k) \\ &= (1 + (-1)^{q-k})P(k, q-1-2k). \end{aligned}$$

□

Corollary 2.2.3. *The kernel of the cellular differential* ∂_q^{cell} *is generated by the sets* $\{P(k, q-2k) \,|\, 2 \not| (q-k)\}$ *and* $\{r \cdot P(k, q-2k) \,|\, 2 \cdot r = 0,\ 2|(q-k)\}$ *as long as* q *is not divisible by 4. Otherwise,* $P(q/2, 0)$ *lies additionally in the kernel. The image is generated by* $\{2 \cdot P(k, q-2k) \,|\, 0 \leq k \leq \lfloor q/2 \rfloor, 2 \not| q-k\}$.

Note that $P(q/2, 0)$ is never a cellular boundary. Therefore, if q is divisible by 4, its induced homology class does not belong to the torsion subgroup.

Example 2.2.4. Let us apply the Corollary 2.2.3 to the cases $\Lambda \in \{\mathbb{Z},\ \mathbb{Z}_2,\ \mathbb{Z}_8\}$.

$\underline{\Lambda = \mathbb{Z}_2}$: Since $2 = 0$ we have $\partial_q = 0$ for every q. Thus, we get for the homology groups $H_q(BO(2); \mathbb{Z}_2) \cong \mathcal{C}_q^{cell}(BO(2); \mathbb{Z}_2) \cong \mathbb{Z}_2^{\lfloor \frac{q}{2}+1 \rfloor}$

$\underline{\Lambda = \mathbb{Z}}$: Using Corollary 2.2.3, we deduce $H_q(BO(2)) \cong \mathbb{Z}_2^{\lfloor q/4+1 \rfloor}$ if q is not divisible by 4, and $H_q(BO(2)) \cong \mathbb{Z}_2^{\lfloor q/4 \rfloor} \oplus \mathbb{Z}$ if q is. In this case, the free part is generated by $P(q/2, 0)$.

$\underline{\Lambda = \mathbb{Z}_8}$: Again, if q is not divisible by 4, then $H_q(BO(2); \mathbb{Z}_8) \cong \mathbb{Z}_2^{\lfloor q/2+1 \rfloor}$ and $H_q(BO(2); \mathbb{Z}_8) \cong \mathbb{Z}_2^{\lfloor q/4 \rfloor} \oplus \mathbb{Z}_8$ otherwise. The free part is generated by $P(q/2, 0)$. A set of generators for the torsion part is given by the set $\{P(k, q-2k) \,|\, q-k \text{ odd}\} \cup \{4 \cdot P(k, q-2k) \,|\, q-k \text{ even}\}$.

The fibre bundle map π given in Theorem 2.1.3 and its section σ_0 are cellular maps. The induced maps on the cellular complexes are given by

$$\mathcal{C}^{cell}(\pi) \colon P(k,l) \mapsto \mathbb{R}P^l \text{ and } \mathcal{C}^{cell}(\sigma_0) \colon \mathbb{R}P^l \mapsto P(0,l).$$

Together with the computation of the cellular boundary, this observation proves the following lemma.

Lemma 2.2.5. *Let R be an abelian group. The kernel of $\mathcal{C}_q^{cell}(\pi)$ is generated by all cells $P(k, q-2k)$ with $k > 0$ and $\ker H_q(\pi) \subseteq H_*(BO(2); R)$ is generated by all elements of the form $r \cdot P(k, q-2k)$ with $k > 0$ and $r \cdot \partial P(k, q-2k) = 0$.*

Having determined the cellular boundary map of the cellular chain complex, the determination of the cellular coboundary map of the cellular cochain complex becomes a formality and is easily done. Therefore, we will omit the calculations and just give the results.

Let $\varphi_{k,q-2k} \in \mathcal{C}^q_{cell}(BO(2); \Lambda) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}_q^{cell}(BO(2); \mathbb{Z}); \Lambda))$ be *dual* to $P(k, q-2k)$ in the sense that $\varphi_{k,q-2k}(P(k, q-2k)) = \delta_{kl}$. These dual cells form a basis of the cellular cochain groups, and the cellular coboundary δ^q_{cell} is uniquely determined by the images of the cocells. The next results are immediate consequences of the calculations done for the homological case.

Theorem 2.2.6. *For the cellular coboundary, we have*

$$\delta^q(\varphi_{k,q-2k}) = (1 + (-1)^{q+1-k}) \cdot \varphi_{k,q+1-2k}.$$

Corollary 2.2.7.

$$\ker \delta^q = \mathrm{span}_R \left(\{\varphi_{k,q-2k} \,|\, 2|(q-k)\} \cup \{r \cdot \varphi_{k,q-2k} \,|\, 2 \cdot r = 0, 2 \not|(q-k)\} \right),$$
$$\mathrm{im}\, \delta^q = \mathrm{span}_R \{2 \cdot \varphi \,|\, 2|(q-k).\}$$

Example 2.2.8. We apply Corollary 2.2.7 to determine the cohomology groups for the cases $\Lambda \in \{\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_8\}$.

$\underline{\Lambda = \mathbb{Z}_2}$: Since $0 = 2$, we have $\delta^q = 0$, and we derive for the cohomology groups $H^q(BO(2); \mathbb{Z}_2) = \mathcal{C}^q_{cell}(BO(2); \Lambda) \cong \mathbb{Z}_2^{\lfloor \frac{q}{2}+1 \rfloor}$.

$\underline{\Lambda = \mathbb{Z}}$: We have $H^q(BO(2); \mathbb{Z}) \cong \mathbb{Z}_2^{\lfloor \frac{q}{4}+1 \rfloor}$ if q is not divisible by 4. Otherwise, we have $H^q(BO(2); \mathbb{Z}) \cong \mathbb{Z}_2^{\lfloor \frac{q}{4} \rfloor} \oplus \mathbb{Z}$. The free part is given by $\varphi_{q/2,0}$.

$\underline{\Lambda = \mathbb{Z}_8}$: If q is not divisible by 4, we have $H^q(BO(2); \mathbb{Z}_8) \cong \mathbb{Z}_2^{\lfloor \frac{q}{2}+1 \rfloor}$. If q is divisible by 4, then $H^q(BO(2); \mathbb{Z}) \cong \mathbb{Z}_2^{\lfloor \frac{q}{2} \rfloor} \oplus \mathbb{Z}$. The free part is given by $\varphi_{q/2,0}$.

Before we close this section, we note that Theorem 2.2.2 also holds for the CW-complexes $P(m,n)$ without any restrictions because they are subcomplexes. On the other hand, the cellular coboundary has to be properly modified because we might 'run out of cocells'. If we use mod 2 coefficients then the cellular boundary and coboundary is always zero. Therefore, we obtain an analogue of Corollary 2.2.4 and 2.2.7 for the Dold-manifolds.

Theorem 2.2.9. *For $0 \leq q \leq 2m+n$, we have*

$$H_q(P(m,n);\mathbb{Z}_2) \cong \mathbb{Z}_2^{\lfloor \frac{q}{2} \rfloor + 1} \cong H^q(P(m,n);\mathbb{Z}_2),$$

and the inclusion $P(m,n) \hookrightarrow P(m',n')$ induces a monomorphism on homology and an epimorphism on cohomology.

2.3. The multiplicative structure of the cohomology ring

We turn now to the description of the multiplicative structure of the cohomology ring of the Dold-manifolds. These results give, in particular, the ring structure of $H^*(BO(2), \mathbb{Z}_2)$ by passing to the limit. Throughout this entire section we are working with mod 2 coefficients only, so we omit them in the notation.

Crucial for the determination of the multiplicative structure is the observation made in Theorem 2.2.9 that the inclusion $P(m,n) \to P(m',n')$ induces an epimorphism on the cellular cochain complex and on the cohomology. There is also a canonical right inverse given by

$$j\colon H^*(P(m,n)) = \mathcal{C}^*_{cell}(P(m,n)) \to \mathcal{C}^*_{cell}(P(m',n')) = H^*(P(m',n')),$$
$$\varphi_{k,l} \mapsto \varphi_{k,l},$$

where $k \le m$ and $l \le n$. This right inverse allows us to interpret $H^*(P(m,n))$ as a subvector space of $H^*(P(m',n'))$.

One easily sees that every cocell $\varphi_{k,l} \in H^q(P(m,n))$ must already lie in $H^q(P(m-1,n))$ or $H^q(P(m,n-1))$ if $q = 2k+l < 2m+n$. Therefore, every element in $H^q(P(m,n))$ is uniquely determined by its images under the homomorphism induced by the inclusions $\iota_1\colon P(m-1,n) \hookrightarrow P(m,n)$ and $\iota_2\colon P(m,n-1) \hookrightarrow P(m,n)$.

Using these informations, we are able to prove inductively the following theorem about the ring structure of the cohomology.

Theorem 2.3.1. *For every $m,n \ge 0$, the assignment $\varphi_{1,0} \mapsto c$ and $\varphi_{0,1} \mapsto d$ extends to an isomorphism of graded algebras*

$$H^*(P(m,n)) \cong \mathbb{Z}_2[c,d]/\left\langle c^{m+1}, d^{n+1}\right\rangle.$$

In particular,

$$H^*(BO(2)) \cong \mathbb{Z}_2[c,d].$$

Proof. The statement is clear for $m = n = 0$, so let us assume that the theorem is correct for all pairs $(m,n) \in \mathbb{N}_0^2$ with $m+n \le N$.

For a pair (m,n) with $m+n = N+1$, we conclude linear independency of $\{\varphi_{1,0}^k \cup \varphi_{0,1}^l \mid 2k+l < 2m+n\}$ from the induction hypothesis. Indeed, if one of these elements, say $\varphi_{1,0}^\alpha \cup \varphi_{0,1}^\beta$, is representable through a linear combination of the others, so is $\iota_j^*(\varphi_{1,0}^\alpha \cup \varphi_{0,1}^\beta)$. But this contradicts the induction hypothesis. Hence, the theorem is already proven for all degrees $< 2m+n$. It remains to show that $\varphi_{1,0}^m \cup \varphi_{0,1}^n \neq 0$. But $P(m,n)$ is a manifold, so we obtain, for $n > 0$, from the cup-cap relation and Poincaré duality [tD08, p.443] the relation

$$\begin{aligned}\left(\varphi_{1,0}^m \cup \varphi_{0,1}^{n-1} \cup \varphi_{0,1}\right) \cap [P(m,n)] &= (\varphi_{1,0}^m \cup \varphi_{0,1}^{n-1}) \cap \varphi_{0,1} \cap [P(m,n)] \\ &\overset{\text{PD}}{=} (\varphi_{1,0}^m \cup \varphi_{0,1}^{n-1}) \cap [P(m,n-1)] \\ &\overset{\text{IV}}{=} 1,\end{aligned}$$

and for $m > 0$, the relation

$$\begin{aligned}\varphi_{1,0}^m \cap [P(m,0)] &= \varphi_{1,0}^m \cap \varphi_{1,0} \cap [P(m,0)] \\ &= \varphi_{1,0}^{m-1} \cap [P(m-1,0)] \overset{\text{IV}}{=} 1.\end{aligned}$$

In any case, these calculations show that $\varphi_{1,0}^m \cup \varphi_{0,1}^n$ is never zero, and the first part of the theorem is therefore proven.

The second part follows from

$$H^*(BO(2)) \cong \lim_{\leftarrow} H^*(P(m,n)) \cong \lim_{\leftarrow} \mathbb{Z}_2[c,d]/\langle c^{m+1}, d^{n+1}\rangle \cong \mathbb{Z}_2[c,d].$$

□

As an application, we are going to show that every proper inclusion $P(k,l) \subseteq P(m,n)$ is not a retract unless $k = l = 0$.

Corollary 2.3.2. *For every pair (k,l) with $0 < k+l$, the inclusion $P(k,l) \subseteq P(m,n)$ has no retract. In particular, the right inverse of ι^* does not arise from a continuous map.*

Proof. Let $r\colon P(m,n) \to P(k,l)$ be a retract of ι. Let c, d be the generators of $H^*(P(m,n))$ as in Theorem 2.3.1 and c', d' the generators for $H^*(P(k,l))$. Since $k + l > 0$, one of them must be non zero, say c'. From $r \circ \iota = \mathrm{id}$ we conclude

$$c^m = (r^*(\iota^*(c)))^m = (r^*(c'))^m = r^*\left((c')^m\right) = r^*(0) = 0,$$

which contradicts the previous theorem. □

2.4. The total Stiefel-Whitney class of the Dold-Manifolds

The aim of this section is to calculate the total Stiefel-Whitney class of the Dold-manifolds. As in the previous section mod 2 coefficients are understood.

Recall from Theorem 2.1.3 that we have a fibre bundle

$$\mathbb{C}P^m \hookrightarrow P(m,n) \overset{p}{\twoheadrightarrow} \mathbb{R}P^n, \qquad \sigma_0\colon \mathbb{R}P^n \dashrightarrow P(m,n)$$

and from the previous section that $H^*(P(m,n)) \cong \mathbb{Z}_2[c,d]/\langle c^{m+1}, d^{n+1}\rangle$.

The remainder of this section is reserved for the proof of the next formula.

Theorem 2.4.1. *The total Stiefel Whitney class of $P(m,n)$ satisfies the relation*

$$w(P(m,n)) = (1+d)^n(1+c+d)^{m+1} \in H^*(P(m,n)).$$

In particular,

$$w_1(P(m,n)) = (m+n+1)\cdot d$$
$$w_2(P(m,n)) = (m+1)\cdot c + \left(\frac{m(m+1)}{2} + \frac{n(n+1)}{2}\right)\cdot d^2.$$

Simple calculations imply the following corollary, which tells us precisely which Dold-manifolds carry a Pin^- structure.

Corollary 2.4.2. *$P(0,n)$ is a Pin^- manifold if and only if $n \equiv 1,2 \bmod 4$. $P(m,0)$ and $P(m,1)$ are Pin^- manifolds if and only if $m \equiv 1 \bmod 2$. For $m > 0$ and $n > 1$ the manifold $P(m,n)$ carries a Pin^- structure if and only if $m \equiv 1 \bmod 2$ and*

$$n \equiv \begin{cases} 1,2 \bmod 4, & \text{if } m \equiv 1 \bmod 4 \\ 0,3 \bmod 4, & \text{if } m \equiv 3 \bmod 4. \end{cases}$$

We will verify the formula for the Stiefel-Whitney class by induction over m and n. In order to do this, we need two auxiliary lemmas for the reduction.

Lemma 2.4.3. *Let $\gamma_{n-1} \to \mathbb{R}P^{n-1}$ be the tautological line bundle over $\mathbb{R}P^{n-1}$. The normal bundle $\nu := \nu\,(P(m,n-1) \hookrightarrow P(m,n))$ satisfies*

$$\nu \cong p^*\gamma_{n-1}$$

Proof. Let $\kappa\colon \mathbb{C}P^m \times S^n \overset{2:1}{\longrightarrow} P(m,n)$ the canonical projection, and consider the bundle map

$$p^*\gamma_{n-1} \to \nu \subseteq TP(m,n)$$
$$([z,x],v) \mapsto [t \mapsto \kappa\,(p, (\cos(\langle x, t\cdot v\rangle))\cdot x, \sin(\langle x, t\cdot v\rangle))],$$

where $[t \mapsto \kappa(\dots)]$ is the tangent vector at the point $[z,x] \in P(m,n-1) \subseteq P(m,n)$ represented by the curve $t \mapsto \kappa(\dots)$. This map is well defined, because it does not depend on the choice of the representative of $[z,x]$, and it maps indeed into the normal bundle because the curve $t \mapsto (p, (\cos(\langle x, t\cdot v\rangle)) \cdot x, \sin(\langle x, t \cdot v\rangle)))$ represents an element in the normal bundle of $S^{n-1} \subseteq S^n$. Furthermore, this map is fibre-wise linear and obviously not the zero map, therefore a fibre-wise isomorphism since both fibres are one-dimensional vector spaces. Additionally, it induces the identity on the base space. Thus, the given map is a vector bundle isomorphism. □

Lemma 2.4.4. *The normal bundle* $\nu := \nu(P(m-1,n) \hookrightarrow P(m,n))$ *has the total Stiefel-Whintey class*

$$w(\nu) = 1 + c + d.$$

Proof. Since $H^1(P(m,n)) = \operatorname{span}_{\mathbb{Z}_2}\{d\}$, $H^2(P(m,n)) = \operatorname{span}_{\mathbb{Z}_2}\{c, d^2\}$, and ν is a vector bundle of rank 2, its total Stiefel-Whitney class can be completely detected by its restriction onto fibre and base space of $P(m,n)$, considered as a $\mathbb{C}P^m$-fibre bundle over $\mathbb{R}P^n$.

Firstly, we have

$$\sigma_0^* w(\nu) = w(\sigma_0^*\nu) = w(\varepsilon \oplus \gamma_{n-1}) = 1 + d \in H^*(\mathbb{R}P^n)$$

because there is a vector bundle isomorphism given by

$$\begin{aligned} &\varepsilon \oplus \gamma_{n-1} \to \sigma_0^*\nu. \\ &\quad (v_1, v_2) \mapsto \left[t \mapsto (\sigma_0 \circ \pi(v_1), [1:0:\cdots:0:t\cdot(v_1 + \mathrm{i}\langle v_2, x\rangle)])\,, x\right]. \end{aligned}$$

Here, $\mathrm{pr}\colon \varepsilon \to \mathbb{R}P^n$ denotes the trivial line bundle, and $[t \mapsto \dots]$ denotes a tangent vector at the point $\sigma_0(\mathrm{pr}(v_1))$. Since the curve

$$t \mapsto [1:0:\cdots:0:t\cdot(v_1 + \mathrm{i}\langle v_2, x\rangle)]$$

represents an element in $\nu(\mathbb{C}P^{m-1} \hookrightarrow \mathbb{C}P^m)$ our map indeed values in $\sigma_0^*\nu$.

Because of the commutativity of

$$\begin{array}{ccc} \mathbb{C}P^{m-1} & \xrightarrow{\text{incl}} & \mathbb{C}P^{m-1} \times S^n \\ & \searrow^{\iota} & \downarrow \kappa \\ & & P(m-1,n) \end{array}$$

and that κ is a submersion, we have

$$\begin{aligned} \iota^* w(\nu) &= w(\iota^*\nu) = w((\kappa \circ \iota)^*\nu) \\ &= w(\mathrm{incl}^*\kappa^*\nu) \\ &= w\left(\mathrm{incl}^*\nu(\mathbb{C}P^{m-1} \times S^n \hookrightarrow \mathbb{C}P^m \times S^n)\right) \\ &= w\left(\nu(\mathbb{C}P^{m-1} \hookrightarrow \mathbb{C}P^m)\right) \\ &= w(\tau_{m-1}^\vee) = c \in H^2(\mathbb{C}P^m). \end{aligned}$$

Here, $\tau_{m-1} \to \mathbb{C}P^{m-1}$ denotes the canonical (complex) line bundle. To see that $(\nu(\mathbb{C}P^{m-1} \hookrightarrow \mathbb{C}P^m))$ is isomorphic to $\tau_{m-1}^\vee$, the dual bundle of τ_{m-1}, recall that $T\mathbb{C}P^m = \operatorname{Hom}(\tau_m, \tau_m^\perp)$, where $\tau_m^\perp$ is the orthogonal complement of τ_m in ε^{m+1} with respect to the standard hermitian metric [Mil74, p.169]. Consequently,

$$\begin{aligned} T\mathbb{C}P^m|_{\mathbb{C}P^{m-1}} &= \mathrm{Hom}(\tau_m, \tau_m^\perp)|_{\mathbb{C}P^{m-1}} = \mathrm{Hom}(\tau_m|_{\mathbb{C}P^{m-1}}, \tau_m^\perp|_{\mathbb{C}P^{m-1}}) \\ &= \mathrm{Hom}(\tau_{m-1}, \tau_{m-1}^\perp \oplus \varepsilon) \\ &= \mathrm{Hom}(\tau_{m-1}, \tau_{m-1}^\perp) \oplus \mathrm{Hom}(\tau_{m-1}, \varepsilon) \\ &= T\mathbb{C}P^{m-1} \oplus \tau_{m-1}^\vee. \end{aligned}$$

Since all elements of $H^q(P(m,n))$ are completely determined by the images of ι^* and σ^* if $q \leq 2$, the claim follows. □

Now, we are going to prove the main theorem of this section.

Proof of Theorem 2.4.1. For $m = n = 0$, the statement is obviously true. So, let us assume that the theorem is already proven for all m and n with $m + n \leq N$.

Note that all $w_k(P(m,n))$ with $k \leq 2(m-1) + n + 1$ can be completely described by restricting them to $P(m-1,n) \overset{\iota_1}{\hookrightarrow} P(m,n)$ and $P(m,n-1) \overset{\iota_2}{\hookrightarrow} P(m,n)$.

Now, with a slight abuse of notation we get, using Lemma 2.4.3,

$$\begin{aligned} \iota_1^* w(TP(m,n)) &= w(\iota_1^* TP(m,n)) \\ &= w\left(TP(m-1,n) \oplus \nu\left(P(m-1,n) \hookrightarrow P(m,n)\right)\right) \\ &= w(TP(m-1,n)) \cdot w(\nu) \\ &= w(TP(m-1,n)) \cdot (1+c+d) \\ &\overset{ass.}{=} (1+d)^n(1+c+d)^m(1+c+d) \\ &= (1+d)^n(1+c+d)^{m+1} \in H^*(P(m-1,n)) \\ &= \iota_1^*(\underbrace{(1+d)^n(1+c+d)^{m+1}}_{\in H^*(P(m,n))}), \end{aligned}$$

and analogously, but applying Lemma 2.4.3 instead, we get

$$\begin{aligned} \iota_2^* w(TP(m,n)) &= w(\iota_2 TP(m,n)) \\ &= w\left(TP(m,n-1) \oplus \nu(P(m,n-1) \hookrightarrow P(m,n))\right) \\ &= w(TP(m-1,n)) \cdot w(\nu(\dots)) \\ &= w(TP(m,n-1))(1+d) \\ &\overset{ass.}{=} (1+d)^{n-1}(1+c+d)^{m+1}(1+d) \\ &= (1+d)^n(1+c+d)^{m+1} \in H^*(P(m,n-1)) \\ &= \iota_2^*(\underbrace{(1+d)^n(1+c+d)^{m+1}}_{\in H^*(P(m,n))}). \end{aligned}$$

Together, both computations show that

$$w_k(P(m,n)) = \left((1+d)^n(1+c+d)^{m+1}\right)_{(k)}$$

for every $k \leq 2m + n - 1$. It remains to verify the equality

$$w_{2m+n}(P(m,n)) = \left((1+d)^n(1+c+d)^{m+1}\right)_{(2m+n)} = (m+1)(n+1)c^m d^n.$$

But it is known (c.f [Mil74, Corollary 11.12]) that the top Stiefel-Whitney class applied to the fundamental class gives Euler characteristic modulo 2. Therefore, we derive

$$\begin{aligned} w_{2m+n}(P(m,n)) \cap [P(m,n)] &\equiv \chi(P(m,n)) \mod 2 \\ &= \sum_{r \leq m,\ s \leq n} (-1)^{\dim P(r,s)} \mod 2 \\ &= \sum_{r \leq m,\ s \leq n} (-1)^{2r}(-1)^s \mod 2 \\ &= (m+1) \cdot \sum_{0 \leq s \leq n} (-1)^s \mod 2 \\ &\equiv (m+1)(n+1) \mod 2. \end{aligned}$$

So, the desired equality follows from this numerical result and Poincaré duality. □

2.5. Determination of the ordinary bordism groups

We are going to apply the results from the previous sections to determine the ordinary bordism groups of $B\mathrm{O}(2)$. It will turn out that the ordinary bordism groups are uniquely determined by mod 2 cohomology and the coefficient groups $\Omega_*^{\mathrm{O}}(\mathrm{pt})$. Again, throughout this section we only work in mod 2 coefficients and omit mentioning this in the notation. A precise statement is given in the next theorem. The rest of this chapter is devoted to its proof.

Theorem 2.5.1. *The $\mathbb{Z}_2$-linear extension of the assignment*

$$[P(m,n)] \otimes [M] \mapsto [P(m,n) \times M, \mathrm{pr}_1]$$

defines an isomorphism $H^*(B\mathrm{O}(2)) \otimes_{\mathbb{Z}_2} \Omega_*^{\mathrm{O}}(\mathrm{pt}) \to \Omega_*^{\mathrm{O}}(B\mathrm{O}(2))$ of $\Omega_*^{\mathrm{O}}(\mathrm{pt})$ modules.

The ingredients of the proof are the ring structure of $H^*(P(m,n))$ and the Theorem of Pontrjagin and Thom that classifies the elements of $\Omega_*^{\mathrm{O}}(\mathrm{pt})$ in terms of Stiefel-Whitney numbers, see [Sto15, p.95]. Putting these informations together, we construct for every element on the left-hand-side a general Stiefel-Whitney number giving 1 by applying it to the chosen element. This implies injectivity, and surjectivity will then follow from a counting argument.

Proof. First, observe that the map in the theorem is well-defined and already linear in the second component. Indeed, let $[P(m,n)] \in H_{2m+n}(B\mathrm{O}(2))$ be the homology class generated by $P(m,n)$ and M_1 bordant to M_2 via B. Hence, the singular manifold $(P(m,n) \times (M_1 \sqcup M_2), \mathrm{pr}_1)$ is bounded by $(P(m,n) \times B, \mathrm{pr}_1)$. Also the $\Omega_*^{\mathrm{O}}(\mathrm{pt})$-equivariance is obvious.

Now, let $\xi \in H_*(B\mathrm{O}(2); \mathbb{Z}_2) \otimes \Omega_*^{\mathrm{O}}(\mathrm{pt})$ be a non-zero element. It can be written as a finite linear combination

$$\xi = \sum [P(k,l)] \otimes [M_{(k,l)}].$$

Pick a summand $[P(m,n)] \otimes [M_{(m,n)}]$ such that $2m+n$ is maximal over all summands in the given linear combination of ξ. (We do not require that this summand is unique.) Choose the general Stiefel-Whitney class

$$\begin{aligned}
&\big(\mathrm{pr}_1^*(x^m y^n) \cup w^I(P(m,n)\big) \times M_{(m,n)}) \\
&= c^m d^n \cup w^I(P(m,n) \times M_{(m,n)}) \in H^*(P(m,n) \times M_{(m,n)}),
\end{aligned}$$

where we identify $H^*(B\mathrm{O}(2))$ with $\mathbb{Z}_2[x,y]$ and $H^*(P(m,n))$ with the truncated polynomial algebra $\mathbb{Z}_2[c,d]/\langle c^{m+1}, d^{n+1}\rangle$. The term w^I is a product of Stiefel-Whitney classes associated to the partition $I = (i_1, \dots, i_k)$ of dimension of $M_{(m,n)}$, in formula $w^I = \prod_j w^{i_j}$, such that $w^I(M_{(m,n)}) \cap [M_{(m,n)}] = 1$. Such a partition exists by a theorem of Thom [Sto15, p.95].

We already calculated that $\mathrm{pr}_1 \colon P(k,l) \times M \to P(k,l) \hookrightarrow B\mathrm{O}(2)$ satisfies

$$\mathrm{pr}_1^*(x^m y^n) \neq 0 \Leftrightarrow k \leq m \text{ and } l \leq n.$$

Since m and n are chosen to be maximal in the linear combination of ξ, we obtain for the generalised Stiefel-Whitney class of the image of ξ under the assignment, which is given by

$$\left[\bigsqcup_{(k,l)} P(k,l) \times M_{(k,l)}, \bigsqcup \mathrm{pr}_1^{(k,l)}\right],$$

the formula

$$\begin{aligned}
&\left(\bigsqcup \mathrm{pr}_1^{(k,l)}\right)^* (x^m y^n) \cup w^I \left(\bigsqcup_{(k,l)} P(k,l) \times M_{(k,l)}\right) \\
&= \mathrm{pr}_1^{(m,n)}(x^m y^n) \cup w^I(P(m,n) \times M_{(m,n)}) \\
&= \mathrm{pr}_1^{(m,n)}(x^m y^n) \cup w^I(M_{(m,n)}).
\end{aligned}$$

Here, the last equality follows from degree reasons. Indeed, the k-th Stiefel Whitney class of $P(m,n) \times M_{(m,n)}$ decomposes into

$$w_k(P(m,n) \times M_{(m,n)}) = \sum_{\alpha=0}^{k} w_\alpha(P(m,n)) \times w_{k-\alpha}(M_{(m,n)}).$$

Since $\mathrm{pr}_1^{(m,n)}(x^m y^n)$ is of maximal degree in $H^*(P(m,n))$, any summand having a factor $w_\alpha(\dots)$ with $\alpha > 0$ must vanish, and the last equation follows. But

$$\begin{aligned}
&\left(\mathrm{pr}_1^{(m,n)}(x^m y^n) \cup w^I(M_{(m,n)})\right) \cap [P(m,n) \times M_{(m,n)}] \\
&= \mathrm{pr}_1^{(m,n)}(P(m,n)) \cap \left(w^I(M_{(m,n)}) \cap [P(m,n) \times M_{(m,n)}]\right) \\
&= \mathrm{pr}_1^*(x^m y^n) \cap [P(m,n)] = 1,
\end{aligned}$$

so the image of ξ is not zero by the theorem of Pontrjagin and Thom. Therefore, the map is injective. Consequently, $\Omega_k^{\mathrm{O}}(B\mathrm{O}(2))$ contains more elements than $\bigoplus_{\alpha+\beta=k} H_\alpha(B\mathrm{O}(2)) \otimes_{\mathbb{Z}_2} \Omega_\beta^{\mathrm{O}}(\mathrm{pt})$. Surjectivity can be deduced from the Atiyah Hirzebruch spectral sequence. For every $k \geq 0$, we have

$$\begin{aligned}
\#\Omega_k^{\mathrm{O}}(B\mathrm{O}(2)) &= \# \bigoplus E^\infty_{\alpha+\beta=k} \leq \# \bigoplus_{\alpha+\beta=k} E^2_{\alpha,\beta} \\
&= \# \bigoplus_{\alpha+\beta=k} H_\alpha(B\mathrm{O}(2), \Omega_\beta^{\mathrm{O}}(\mathrm{pt})) \\
&= \# \left(H_*(B\mathrm{O}(2)) \otimes_{\mathbb{Z}_2} \Omega_*^{\mathrm{O}}(\mathrm{pt})\right)_{(k)} < \infty.
\end{aligned}$$

Now, surjectivity follows from injectivity. □

This theorem is actually true for any CW-pair (X, A), see [Sto15, p.108ff]. The proof given there uses essentially the same strategy, but it needs more theory to show that any element in $H_*(X, A)$ can be represented by some element in $\Omega_*^{\mathrm{O}}(X, A)$. In our case, where the closed cells are submanifolds, we avoid this problem completely and are able to give an explicit description of this isomorphism. Note, that this theorem also applies to $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$ as well. Furthermore, this theorem implies that the AHSS for ordinary bordism has only vanishing higher differentials, meaning $d^r_{p,q} = 0$ for every $r \geq 2$. Thus, $E^\infty_{p,q} = E^2_{p,q}$.

Corollary 2.5.2. *The inverse of the map in Theorem 2.5.1 sends $F_{p,q}$ to $\bigoplus_{(\alpha,\beta)\in I_{(p,q)}} H_\alpha(BO(2), \Omega^{O}_\beta(\mathrm{pt}))$, where $I_{(p,q)} := \{(\alpha,\beta) \mid \alpha+\beta = p+q, \alpha \leq p\}$.*

Proof. Since unoriented bordism is a $\mathbb{Z}_2$-vector space, the corollary follows from the solution of the extension problem:

$$\begin{aligned} F_{p,q} &= F_{p-1,q+1} \oplus E^\infty_{p,q} \\ &= F_{p-1,q+1} \oplus E^2_{p,q} \\ &= F_{p-1,q+1} \oplus H_p(BO(2), \Omega^{O}_q(\mathrm{pt})). \end{aligned}$$

The stated formula follows inductively. □

Chapter 3

Calculations of the main results

Having developed the necessary theory, we are in the position to calculate the Pin^- bordism groups of the classifying spaces $B\mathbb{Z}_2$, $B\mathrm{SO}(2)$, and $B\mathrm{O}(2)$.

The method of our choice will be the Atiyah-Hirzebruch spectral sequence because this spectral sequence is very close to the homology theory it approximates, and allows us therefore to find geometric representatives for the generators of these groups.

The structure of this chapter is fairly simple. We start in the first section with the easiest case, namely $\Omega_*^{\mathrm{Pin}^-}(B\mathrm{SO}(2))$. Then we turn our attention to the calculation of $\Omega_*^{\mathrm{Pin}^-}(B\mathbb{Z}_2)$, which will be done in the second section. Finally, the hardest case, which is the determination of $\Omega_*^{\mathrm{Pin}^-}(B\mathrm{O}(2))$, will be done in the last section of this chapter.

3.1. Calculation of $\Omega_*^{\mathrm{Pin}^-}(\mathbb{C}P^\infty)$

Using the Pin^- bordism coefficients listed in Corollary 1.5.2 and the cellular structure of $\mathbb{C}P^\infty$, we are in the position to calculate the Pin^- bordism groups of $\mathbb{C}P^\infty$ up to degree 4.

The results of the involved spectral-sequence-calculations are summarised in the next theorem. Its proof is the aim of this section.

Theorem 3.1.1. *For the* Pin^- *bordism groups of* $\mathbb{C}P^\infty$ *up to degree* 4*, we have the following results:*

$$\begin{aligned}
\Omega_0^{\mathrm{Pin}^-}(\mathbb{C}P^\infty, \mathrm{pt}) &\cong 0,\\
\Omega_1^{\mathrm{Pin}^-}(\mathbb{C}P^\infty, \mathrm{pt}) &\cong 0,\\
\Omega_2^{\mathrm{Pin}^-}(\mathbb{C}P^\infty, \mathrm{pt}) &\cong \mathbb{Z}_2,\\
\Omega_3^{\mathrm{Pin}^-}(\mathbb{C}P^\infty, \mathrm{pt}) &\cong 0,\\
\Omega_4^{\mathrm{Pin}^-}(\mathbb{C}P^\infty, \mathrm{pt}) &\cong \mathbb{Z}_4.
\end{aligned}$$

The group $\Omega_2^{\mathrm{Pin}^-}(\mathbb{C}P^\infty, \mathrm{pt})$ *is generated by* $[\mathbb{C}P^1, \mathrm{incl}]$ *and* $\Omega_4^{\mathrm{Pin}^-}(\mathbb{C}P^\infty, \mathrm{pt})$ *has* $[\mathbb{C}P^1 \times \mathbb{R}P^2, \mathrm{pr}_1]$ *as generator.*

Let us start with a (partial) description of the action of $[S^1_{\mathrm{Lie}}] \in \Omega_1^{\mathrm{Spin}}(\mathrm{pt})$ on $E^2_{p,q}$. Since the multiplication with $[S^1_{\mathrm{Lie}}]$ induces an isomorphism from $\Omega_0^{\mathrm{Pin}^-}(\mathrm{pt})$ to $\Omega_1^{\mathrm{Pin}^-}(\mathrm{pt})$, it also induces an isomorphism $E^2_{p,0} \to E^2_{p,1}$. We know that $0 \neq [S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}] \in \Omega_2^{\mathrm{Pin}^-}(\mathrm{pt})$ is the unique element in $\mathbb{Z}_8$ of order 2, see Corollary 1.5.11, so the multiplication with $[S^1_{\mathrm{Lie}}]$ give rise to an injective map $E^2_{p,1} \to E^2_{p,2}$.

Forgetting the Pin^- structure gives the following homomorphisms on the coefficient groups

$$\begin{aligned}
\Omega_0^{\mathrm{Pin}^-}(\mathrm{pt}) &\xrightarrow{\cong} \Omega_0^{\mathrm{O}}(\mathrm{pt}),\\
\Omega_1^{\mathrm{Pin}^-}(\mathrm{pt}) &\xrightarrow{\cdot 0} \Omega_1^{\mathrm{O}}(\mathrm{pt}),\\
\Omega_2^{\mathrm{Pin}^-}(\mathrm{pt}) &\twoheadrightarrow \Omega_2^{\mathrm{O}}(\mathrm{pt}),
\end{aligned}$$

which carry over to the spectral sequence

$$\begin{aligned}
{}^{\mathrm{Pin}}E^2_{p,0} &\xrightarrow{\cong} {}^{\mathrm{O}}E^2_{p,0},\\
{}^{\mathrm{Pin}}E^2_{p,1} &\xrightarrow{\cdot 0} {}^{\mathrm{O}}E^2_{p,1},\\
{}^{\mathrm{Pin}}E^2_{p,2} &\twoheadrightarrow {}^{\mathrm{O}}E^2_{p,2}.
\end{aligned}$$

We have seen that the cellular differentials of the standard cellular chain complex of $\mathbb{C}P^\infty$ are always zero. Thus, $E^2_{p,q} \cong \Omega_q^{\mathrm{Pin}^-}(\mathrm{pt})$ if p is even and non-negative. Otherwise, the groups vanish. The part of our interest is presented in Figure 3.1.

4	0	0	0	0	0	0
3	0	0	0	0	0	0
2	$\mathbb{Z}_8$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_8$	0
1	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
	0	1	2	3	4	5

Figure 3.1. The second page $E^2_{p,q}$ for $p \leq 5$ and $q \leq 4$ of the Atiyah-Hirzebruch spectral sequence for $\Omega_*^{\mathrm{Pin}^-}(\mathbb{C}P^\infty)$.

Now, we calculate the non-trivial differentials of our interest. The results are listed in the next theorem.

Theorem 3.1.2.

$$d^2_{2,0} = 0, \qquad d^2_{4,0} \neq 0,$$
$$d^2_{2,1} = 0, \qquad d^2_{4,1} \neq 0.$$

Proof. Every differential of the form $d^2_{2,q}$ is zero because its target is space lies in the column indexed by $p = 0$. By equivariance, we have

$$\begin{array}{ccc}
E^2_{p,0} & \xrightarrow{d^2_{p,0}} & E^2_{p-2,q+1} \\
{\scriptstyle \cdot\times[S^1_{\mathrm{Lie}}]}\big\downarrow\cong & & \big\downarrow{\scriptstyle \cdot\times[S^1_{\mathrm{Lie}}]} \\
E^2_{p,1} & \xrightarrow{d^2_{p,1}} & E^2_{p-2,2},
\end{array}$$

4	0	0	0	0	0	0
3	0	0	0	0	0	0
2	$\mathbb{Z}_8$	0	$\mathbb{Z}_4$	0	?	0
1	$\mathbb{Z}_2$	0	0	0	0	0
0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0
	0	1	2	3	4	5

Figure 3.2. third page $E^3_{p,q}$ for $p \leq 5$ and $q \leq 4$ of the Atiyah-Hirzebruch spectral sequence of $\Omega_*^{\mathrm{Pin}^-}(\mathbb{C}P^\infty)$.

and $d^2_{p,1}$ is therefore uniquely determined by $d^2_{p,0}$. In particular, if $d^2_{4,0}$ does not vanish, so does not $d^2_{4,1}$. Having this observation in mind, it remains to show that $d^2_{4,0}$ is not the zero map. So, let us assume the contrary. Note that $d^3_{4,0}$ is zero because its target space is, and that $d^4_{4,0}$ is zero because it maps into the column indexed by $p = 0$. We conclude that $E^\infty_{4,0} = E^2_{4,0} \cong \mathbb{Z}_2$. It follows that forgetting the Pin^- structure induces an isomorphism

$$^{\mathrm{Pin}^-}E^\infty_{4,0} = {}^{\mathrm{Pin}^-}E^2_{4,0} \to {}^{\mathrm{O}}E^2_{4,0} = {}^{\mathrm{O}}E^\infty_{4,0}.$$

So, there exists an element in ${}^{\mathrm{O}}F_{4,0} \setminus {}^{\mathrm{O}}F_{3,1}$ that can be represented by a singular Pin^- manifold. But we will rule this out using characteristic numbers.

Any element in ${}^{\mathrm{O}}F_{4,0} \setminus {}^{\mathrm{O}}F_{3,1}$ is either given by $[\mathbb{C}P^2, \mathrm{incl}] + [M^4, \mathrm{const}]$ or by $[\mathbb{C}P^2, \mathrm{incl}] + [\mathbb{C}P^1 \times \mathbb{R}P^2, \mathrm{pr}_1] + [M^4, \mathrm{const}]$. Since $\mathbb{C}P^1 \times \mathbb{R}P^2$ is already a Pin^- manifold, we only have to check that $[\mathbb{C}P^2, \mathrm{incl}] + [M^4, \mathrm{const}]$ cannot be represented by a Pin^- manifold.

Identify $H^*(\mathbb{C}P^\infty; \mathbb{Z}_2)$ with $\mathbb{Z}_2[x]$ and $H^*(\mathbb{C}P^2; \mathbb{Z}_2)$ with $\mathbb{Z}_2[a]/\langle a^3 \rangle$. Using this notation we get for the general Stiefel-Whitney class

$$\begin{aligned}(\mathrm{incl} \sqcup \mathrm{const})^*(x) \cup (w_2 - w_1^2)(\mathbb{C}P^2 \sqcup M^4) &= \mathrm{incl}^*(x) \cup (w_2 - w_1^2)(\mathbb{C}P^2) \\ &= \mathrm{incl}^*(x) \cup w_2(\mathbb{C}P^2) \\ &= a^2 \neq 0.\end{aligned}$$

Poincaré duality implies that the associated generalised Stiefel-Whitney number is not zero. Therefore, $[\mathbb{C}P^2, \mathrm{incl}] + [M^4, \mathrm{const}]$ cannot be represented by a singular Pin^- manifold. We derived the desired contradiction, and conclude $d^2_{4,0} \neq 0$. □

The determination of the differentials gives the (partial) result for the third page listed in Figure 3.2. We conclude $E^3_{p,q} = E^\infty_{p,q}$ for $p + q \leq 5$.

The extension problem is easily solved. The groups in the zero-column $E^\infty_{0,q}$ gives $\Omega_q^{\mathrm{Pin}^-}(\mathrm{pt}) \subseteq \Omega_q^{\mathrm{Pin}^-}(\mathbb{C}P^\infty)$, which always splits off. Therefore, we deduced the isomorphisms given in Theorem 3.1.1.

Lastly, we have to find generators for the groups. Note that $(\mathbb{C}P^1, \mathrm{incl})$ and $(\mathbb{C}P^1 \times \mathbb{R}P^2)$ are singular Pin^- manifolds, which represent non-zero elements in $\Omega_*^{\mathrm{O}}(\mathbb{C}P^\infty)$ by Theorem 2.5.1. Therefore, those singular Pin^- manifolds cannot be Pin^- boundaries. For algebraic reasons, these elements must be generators.

3.2. Calculation of $\Omega_*^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$

We turn now to the calculations of the first five Pin^- bordism groups of $\mathbb{R}P^\infty$. Since these groups sit as split summands in the Pin^- bordism groups of $B\mathrm{O}(2)$, their calculations will be a milestone towards our final goal.

The result of this section are summarised in the following theorem. Its proof is the contest of this section.

Theorem 3.2.1.

$$\begin{aligned}
&\Omega_0^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt}) \cong 0, && \\
&\Omega_1^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt}) \cong \mathbb{Z}_2, && \textit{generated by } [\mathbb{R}P^1, \mathrm{incl}], \\
&\Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt}) \cong \mathbb{Z}_4, && \textit{generated by } [\mathbb{R}P^2, \mathrm{incl}], \\
&\Omega_3^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, && \\
&\Omega_4^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt}) \cong 0. &&
\end{aligned}$$

Generators of $\Omega_3^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt})$ *are* $[\mathbb{R}P^3, \mathrm{incl}]$ *and* $[\mathbb{R}P^1 \times \mathbb{R}P^2, \mathrm{pr}_1]$.

Before we begin with the calculation, let us have a closer look at the structure of this spectral sequence. First, recall that forgetting the orientation

$$\mathrm{FORTGET}\colon \Omega_*^{\mathrm{Spin}}(\mathbb{R}P^\infty) \to \Omega_*^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$$

induces a map between the corresponding Atiyah-Hirzebruch spectral sequences, see Theorem C.0.19. For $q \in \{0,1\}$, this induced map ${}^{\mathrm{Spin}}E^2_{p,q} \hookrightarrow {}^{\mathrm{Pin}^-}E^2_{p,q}$ is always injective. For $q = 2$, the resulting map is injective only if p is even. This follows from the knowledge of the generators of ${}^{G}E^2_{p,q}$ and the induced map on ${}^{G}E^1_{p,q}$, which is given under the correspondence ${}^{G}E^1_{p,q} \cong \mathcal{C}_p^{cell}(\mathbb{R}P^\infty) \otimes \Omega_q^G(\mathrm{pt})$ by

$$\begin{array}{ccc}
{}^{\mathrm{Spin}}E^1_{p,q} & \longrightarrow & {}^{\mathrm{Pin}^-}E^1_{p,q} \\
\downarrow \cong & & \downarrow \\
\mathcal{C}_p^{cell}(\mathbb{R}P^\infty) \otimes \Omega_q^{\mathrm{Spin}}(\mathrm{pt}) & \xrightarrow{\mathrm{id}\otimes\mathrm{FORGET}} & \mathcal{C}_p^{cell}(\mathbb{R}P^\infty) \otimes \Omega_q^{\mathrm{Pin}^-}(\mathrm{pt}).
\end{array}$$

We will do the calculation for the most interesting case $q = 2$ in detail. The other cases can be treated in the same manner, but are much easier. Recall that the group ${}^{\mathrm{Spin}}E^2_{p,2} = H_p(\mathbb{R}P^\infty, \Omega_2^{\mathrm{Spin}}(\mathrm{pt}))$ is generated by

$$\mathbb{R}P^p \otimes [S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}] \in C_p^{cell}(\mathbb{R}P^\infty) \otimes \Omega_2^{\mathrm{Spin}}(\mathrm{pt}),$$

which corresponds to 4 under the identification $C_p^{cell}(\mathbb{R}P^\infty) \otimes \Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}_8$. From the cellular boundary operator, see Theorem 2.2.2, we deduce that $\mathbb{R}P^p \otimes [S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}]$ is a cellular boundary if and only if p is odd.

Using a similar argument, we see that the $\Omega_*^{\mathrm{Spin}}(\mathrm{pt})$-action on ${}^{\mathrm{Pin}^-}E^2_{p,q}$ is non-zero for $q = 0$ and therefore injective. For $q = 1$, this action is non-zero if and only if p is even.

Furthermore, the homomorphism ${}^{\mathrm{Pin}^-}E^2_{p,q} \to {}^{\mathrm{O}}E^2_{p,q}$ is surjective for $q = 0$, zero for $q = 1$. For $q = 2$, it is non-zero if and only if p is odd.

4	0	0	0	0	0	0
3	0	0	0	0	0	0
2	$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	0	1	2	3	4	5

Figure 3.3. The second page $E^2_{p,q}$ for $p \leq 5$ and $q \leq 4$ of the Atiyah-Hirzebruch spectral sequence of $\Omega_*^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$.

The part of the spectral sequence we are interested in is listed in Figure 3.3, and the differentials we are interested in are listed in the next theorem.

Theorem 3.2.2.

$$\begin{array}{lll} d^2_{2,0} = d^2_{2,1} = 0, & d^2_{4,0} \neq 0, & d^2_{5,0} \neq 0, \\ d^2_{3,0} = d^2_{3,1} = 0, & d^2_{4,1} \neq 0, & d^2_{5,1} = 0. \end{array}$$

Proof. The differentials $d^2_{2,0}$ and $d^2_{2,1}$ have their target space in the column indexed by $p = 0$, so they have to be zero-maps. From the commutativity of

$$\begin{array}{ccc} {}^{\mathrm{Spin}}E^2_{3,0} & \xrightarrow{d^2_{3,0}} & {}^{\mathrm{Spin}}E^2_{1,1} \\ \cong\downarrow & & \downarrow\cong \\ {}^{\mathrm{Pin}^-}E^2_{3,0} & \xrightarrow{d^2_{3,0}} & {}^{\mathrm{Pin}^-}E^2_{1,1} \end{array}$$

and Theorem 1.5.6 we deduce $d^2_{3,0} = 0$. Using the same argument, we derive $d^2_{5,0} \neq 0$. From $\Omega_*^{\mathrm{Spin}}(\mathrm{pt})$-equivariance, we get $d^2_{3,1} = 0$ and $d^2_{5,1} = 0$. The determination of $d^2_{4,0}$ is more complicated. First, observe that any differential with target space $E^r_{1,3}$ has to be zero. Indeed, $[\mathbb{R}P^1 \times \mathbb{R}P^2, \mathrm{pr}_1] \in {}^{\mathrm{O}}F_{1,2} \setminus {}^{\mathrm{O}}F_{0,3}$ is represented by a Pin^- manifold and lies therefore in ${}^{\mathrm{Pin}^-}F_{1,2} \setminus {}^{\mathrm{Pin}^-}F_{0,3}$, too. So, $E^\infty_{1,2}$ must be non-zero. From $E^2_{1,2} \cong \mathbb{Z}_2$ follows that every differential into $E^r_{1,2}$ must vanish, in particular $d^3_{4,0}$.

Now, assume $d^2_{4,0} = 0$. Then the previous discussion implies $\mathbb{Z}_2 \cong E^2_{4,0} = E^\infty_{4,0}$. Since forgetting the Pin^- structure ${}^{\mathrm{Pin}^-}E^2_{4,0} \to {}^{\mathrm{O}}E^2_{4,0}$ is an isomorphism, ${}^{\mathrm{Pin}^-}E^\infty_{4,0} \to {}^{\mathrm{O}}E^\infty_{4,0}$ is one, too. Therefore, there is an element ${}^{\mathrm{O}}F_{4,0} \setminus {}^{\mathrm{O}}F_{3,1}$ that can be represented by a singular Pin^- manifold. We use characteristic numbers to rule this out.

By Corollary 2.5.2, any element $\xi \in {}^{\mathrm{O}}F_{4,0} \setminus {}^{\mathrm{O}}F_{3,1}$ can be written as

$$\xi = \begin{cases} [\mathbb{R}P^4, \mathrm{incl}] + [M^4, \mathrm{const}] \\ [\mathbb{R}P^4, \mathrm{incl}] + [\mathbb{R}P^2 \times \mathbb{R}P^2, \mathrm{pr}_1] + [M^4, \mathrm{const}]. \end{cases}$$

Identify $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2)$ with $\mathbb{Z}_2[x]$ and $H^*(\mathbb{R}P^4;\mathbb{Z}_2)$ with $\mathbb{Z}_2[a]/\langle a^5\rangle$. Hence, the generalised Stiefel-Whitney class

$$\begin{aligned}&\big(w_2(\mathbb{R}P^4 \sqcup M^4) - w_1^2(\mathbb{R}P^4 \sqcup M^4)\big) \cup \mathrm{incl}^* x^2\\ &= \big(w_2(\mathbb{R}P^4) - w_1^2(\mathbb{R}P^4)\big) \cup \mathrm{incl}^*(x^2)\\ &= (0 - a^2)\cdot a^2\\ &= a^4 \neq 0\end{aligned}$$

gives by Poincaré duality a non-zero generalised Stiefel-Whitney number. So, in the first case, ξ cannot be represented by a singular Pin^- manifold. In the second case, set $N := \mathbb{R}P^4 \sqcup \mathbb{R}P^2 \times \mathbb{R}P^2 \sqcup M^4$, and identify $H^*(\mathbb{R}P^2 \times \mathbb{R}P^2;\mathbb{Z}_2)$ with $\mathbb{Z}_2[a,b]/\langle a^3, b^3\rangle$ as well as $H^*(\mathbb{R}P^4;\mathbb{Z}_2)$ with $\mathbb{Z}_2[c]/\langle c^5\rangle$. Then, the generalised Stiefel-Whitney number

$$\begin{aligned}&\big(w_2(N) - w_1(N)^2\big) \cup (\mathrm{incl} \cup \mathrm{pr}_1 \cup \mathrm{const})^*(x^2) \cap [N]\\ &= \big(w_2(\mathbb{R}P^4) - w_1(\mathbb{R}P^4)^2 \cup \mathrm{incl}^*(x^2)\big) \cap [\mathbb{R}P^4]\\ &\quad + \Big(\big(w_2((\mathbb{R}P^2)^2) + w_1^2((\mathbb{R}P^2)^2)\big) \cup \mathrm{pr}_1^*(x^2)\Big) \cap [(\mathbb{R}P^2)^2]\\ &= c^4 \cap [\mathbb{R}P^4] + \big((a^2 + ab + b^2 - (a+b)^2) \cup a^2\big) \cap [\mathbb{R}P^2 \times \mathbb{R}P^2]\\ &= c^4 \cap [\mathbb{R}P^4] + a^3 b \cap [\mathbb{R}P^2 \times \mathbb{R}P^2]\\ &= 1 + 0 = 1 \neq 0\end{aligned}$$

shows that the second possible description for ξ cannot be the boundary of a singular Pin^- manifold. Consequently, no element in ${}^{\mathrm{O}}F_{4,0} \setminus {}^{\mathrm{O}}F_{3,1}$ can be hit by a singular Pin^- manifold, and, therefore, $d^2_{4,0}$ must not be the zero map. From this result we deduce $d^2_{4,1} \neq 0$ using $\Omega_*^{\mathrm{Spin}}(\mathrm{pt})$ equivariance. □

The implications for the third page are presented in Figure 3.4.

4	0	0	0	0	0	0
3	0	0	0	0	0	0
2	$\mathbb{Z}_8$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	?	?
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0	?
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0
	0	1	2	3	4	5

Figure 3.4. The third page for $p \leq 5$ and $q \leq 4$ for the Atiyah-Hirzebruch spectral sequence approximating $\Omega_*^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$.

From this table, we deduce $E^3_{p,q} = E^\infty_{p,q}$ for $p+q \leq 4$ because, for every (p,q) with $p+q \leq 4$ and $r \geq 0$, we have $d^r_{p+q} = 0$.

Let us discuss now the extension problem. One easily reads off

$$\begin{aligned}&\Omega_0^{\mathrm{Pin}^-}(\mathbb{R}P^\infty) \cong \mathbb{Z}_2\\ &\Omega_1^{\mathrm{Pin}^-}(\mathbb{R}P^\infty) = \Omega_1^{\mathrm{Pin}^-}(\mathrm{pt}) \oplus \Omega_1^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2\\ &\Omega_4^{\mathrm{Pin}^-}(\mathbb{R}P^\infty) \cong 0.\end{aligned}$$

The generator of $\Omega_0^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$ is just the point. The first summand of $\Omega_1^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$ is generated by $[S^1_{\mathrm{Lie}}]$ as we have shown in Corollary 1.5.2, while the second summand is generated by $[\mathbb{R}P^1, \mathrm{incl}]$ because it generates $E^2_{1,0} = E^\infty_{1,0}$ and lies indeed in $\ker \mathrm{const}_* \cong \Omega_1^{\mathrm{Pin}^-}(\mathbb{R}P^1, \mathrm{pt})$.

The extension problem for $\Omega_3^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$ is also not so hard. Since forgetting the Pin^- structure gives an isomorphism

$$^{\mathrm{Pin}^-}E^\infty_{1,2} \oplus {}^{\mathrm{Pin}^-}E^\infty_{3,0} = {}^{\mathrm{Pin}^-}E^2_{1,2} \oplus {}^{\mathrm{Pin}^-}E^2_{3,0} \xrightarrow{\cong} {}^{\mathrm{O}}E^\infty_{1,2} \oplus {}^{\mathrm{O}}E^\infty_{3,0} = {}^{\mathrm{O}}E^\infty_{1,2} \oplus {}^{\mathrm{O}}E^\infty_{3,0}$$

and this sums are the associated graded module of the corresponding bordism groups in degree three, it already gives an isomorphism between the bordism groups

$$\Omega_3^{\mathrm{Pin}^-}(\mathbb{R}P^\infty) \xrightarrow{\text{FORGET}} \Omega_3^{\mathrm{O}}(\mathbb{R}P^\infty) \cong \mathbb{Z}_2^2.$$

Thus, generators of $\Omega_3^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$ are given by $[\mathbb{R}P^1 \times S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}, \mathrm{pr}_1]$ and $[\mathbb{R}P^3, \mathrm{incl}]$.

Solving the extension problem of $\Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$ heavily relies on the next lemma.

Lemma 3.2.3. *The abelian group* $\Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$ *can be generated by two elements.*

Before we prove the lemma, let us harvest its consequences. The split $\Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty) \cong \Omega_2^{\mathrm{Pin}^-}(\mathrm{pt}) \oplus \Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt})$ and Lemma 3.2.3 imply that the relative group $\Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt})$ must be cyclic. Since the associated graded module $E^\infty_{1,1} \oplus E^\infty_{2,0}$ contains four elements, the relative group is isomorphic to $\mathbb{Z}_4$. So the theorem is proven if Lemma 3.2.3 is.

Proof of Lemma 3.2.3. We will show that any element in $\Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$ lies in the span of $[\mathbb{R}P^2, \mathrm{const}]$ and $[\mathbb{R}P^2, \mathrm{incl}]$.

Note that $[S^1 \times S^1_{\mathrm{Lie}}, \mathrm{pr}_1] \in \Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt})$ generates $E^2_{1,1} = E^\infty_{1,1}$, and it generates therefore a subgroup of order 2. On the other hand, the element $[\mathbb{R}P^2, \mathrm{incl}] - [\mathbb{R}P^2, \mathrm{const}] \in \Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt})$ generates $E^\infty_{2,0}$ because $[\mathbb{R}P^2, \mathrm{incl}]$ does it. By Theorem 1.3.7, we get

$$\begin{aligned}[\mathbb{R}P^2, \mathrm{incl}] - [\mathbb{R}P^2, \mathrm{const}] &= [\mathbb{R}P^2 \# \overline{\mathbb{R}P^2}, \mathrm{incl}\#\mathrm{const}] \\ &= [K, f],\end{aligned}$$

where K is the Klein bottle.

Note that $[K, f]$ lies in $F_{2,0} \backslash F_{1,1}$, but $[S^1 \times S^1_{\mathrm{Lie}}, \mathrm{pr}_1]$ does not. We conclude

$$[S^1 \times S^1_{\mathrm{Lie}}, \mathrm{pr}_1] + [K, f] = [S^1 \times S^1_{\mathrm{Lie}} \# K, \mathrm{pr}_1 \# f] \neq 0.$$

But, by the classification theorem for closed surfaces, we know that

$$\begin{aligned}S^1 \times S^1_{\mathrm{Lie}} \# K &\approx S^1 \times S^1_{\mathrm{Lie}} \# \mathbb{R}P^2 \# \mathbb{R}P^2 \\ &\approx \mathbb{R}P^2_{(1)} \# \mathbb{R}P^2_{(2)} \# \mathbb{R}P^2_{(3)} \# \mathbb{R}P^2_{(4)},\end{aligned}$$

where $\mathbb{R}P^2_{(j)}$ refers to a real projective plane with some appropriate chosen Pin^- structure. Let φ denote the composition of these two Pin^- structure preserving diffeomorphism and set $g := (\mathrm{pr}_1 \# f) \circ \varphi$. Hence,

$$[S^1 \times S^1_{\mathrm{Lie}}, \mathrm{pr}_1] + [K, f] = [\mathbb{R}P^2_{(1)} \# \mathbb{R}P^2_{(2)} \# \mathbb{R}P^2_{(3)} \# \mathbb{R}P^2_{(4)}, g].$$

It was shown in Example 1.3.9 that every Pin^- structure on a connected sum of real projective planes can be obtained from the connected sum procedure by choosing appropriate Pin^- structures on the summands. It remains to show that every map $g\colon \#^k\mathbb{R}P^2 \to \mathbb{R}P^\infty$ can be homotopied into $\# f_j$, where f_j is either homotopic to the constant map or the inclusion. Since $\mathbb{R}P^\infty = K(\mathbb{Z}_2, 1)$, the representability theorem states that

$$\begin{aligned}[X, \mathbb{R}P^\infty] &\to H^1(X, \mathbb{Z}_2)\\ [f] &\mapsto H^1(f)(x)\end{aligned}$$

is bijection [DK01, Theorem 7.22], which is natural with respect to continuous maps. Thus, we can prove this statement using cohomology. Indeed, from the Mayer-Vietoris sequence we get a diagram of isomorphic cohomology groups in mod 2 coefficients

$$\begin{array}{c} H^1(\#_{j=1}^n \mathbb{R}P^2) \\ \cong \big\downarrow \oplus\mathrm{incl}^* \\ H^1(\mathbb{R}P^2\setminus D^2) \oplus \bigoplus_{j=2}^{n-1} H^1(\mathbb{R}P^2 \setminus (D^2 \sqcup D^2)) \oplus H^1(\mathbb{R}P^2 \setminus D^2) \\ \cong \big\uparrow \oplus\mathrm{incl}^* \\ \bigoplus_{j=1}^n H^1(\mathbb{R}P^2). \end{array}$$

The naturality of the bijection now implies that the sets $[\#_{j=1}^n \mathbb{R}P^2, \mathbb{R}P^\infty]$ and $[\bigsqcup_{j=1}^n \mathbb{R}P^2, \mathbb{R}P^\infty]$ have the same number of elements. Since any element in $H^1(\mathbb{R}P^2 \setminus D^2) \oplus \ldots H^1(\mathbb{R}P^2 \setminus D^2)$ is given by some $\oplus\mathrm{incl}^* \#(f_j)$, we deduce from bijectivity that these map build the whole set $[\mathbb{R}P^2, \mathbb{R}P^\infty]$. Moreover, two sums $\#_j f_j^1$ and $\# f_j^2$ are homotopic if and only all their summands f_j^1, f_j^2 are homotopic.

Consequently, the element $[S^1 \times S^1_{\mathrm{Lie}} \# K, \mathrm{pr}_1 \# f]$ lies in the subgroup of $\Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty)$ generated by $[\mathbb{R}P^2, \mathrm{incl}]$ and $[\mathbb{R}P^2, \mathrm{const}]$. Therefore, every element does. □

Corollary 3.2.4. $\Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt})$ *is cyclic.*

Proof. If we assume the contrary, then the set

$$\Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty) = \Omega_2^{\mathrm{Pin}^-}(\mathrm{pt}) \oplus \Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt})$$

must have at least three generators; this contradicts the previous lemma. □

3.3. Calculation of $\Omega_*^{\mathrm{Pin}^-}(B\mathrm{O}(2))$ in low degrees

We are heading to the main results of this thesis, namely the determination of $\Omega_p^{\mathrm{Pin}^-}(B\mathrm{O}(2))$ for $p \leq 4$.

Recall that we have a $\mathbb{C}P^\infty$-fibre bundle structure

$$\mathbb{C}P^\infty \xhookrightarrow{\iota} B\mathrm{O}(2) \xrightarrow{\pi} \mathbb{R}P^\infty, \qquad \sigma\colon \mathbb{R}P^\infty \dashrightarrow B\mathrm{O}(2)$$

on $B\mathrm{O}(2)$ and that the section σ gives a splitting

$$\pi_* \oplus (\mathrm{id} - \sigma_* \circ \pi_*)\colon \Omega_*^{\mathrm{Pin}^-}(B\mathrm{O}(2)) \cong \Omega_*^{\mathrm{Pin}^-}(\mathbb{R}P^\infty) \oplus \ker \pi_*.$$

We made a huge step towards this goal by determining the split summand $\Omega_p^{\mathrm{Pin}^-}(\mathbb{R}P^\infty) \subseteq \Omega_p^{\mathrm{Pin}^-}(B\mathrm{O}(2))$. With the help of the splitting property of the Atiyah-Hirzebruch spectral sequence, see Theorem C.0.21, we will calculate the other summand $\ker \pi_*$. The next theorem summarises the results we are going to achieve.

Theorem 3.3.1. *For the kernel groups we have the following results:*

$$\begin{aligned}
\ker \pi_0 &= 0,\\
\ker \pi_1 &= 0,\\
\ker \pi_2 &= \mathbb{Z}_2,\\
\ker \pi_3 &= \mathbb{Z}_2,\\
\ker \pi_4 &= \mathbb{Z}_2 \oplus \mathbb{Z}_2,
\end{aligned}$$

where $[P(1,0), \mathrm{incl}]$ *generates* $\ker \pi_2$*,* $[P(1,1), \mathrm{incl}]$ *generates* $\ker \pi_3$*, and the singular manifolds* $[P(1,2), \mathrm{incl}]$ *and* $[P(1,0) \times P(0,2), \mathrm{pr}_1]$ *form a basis of* $\ker \pi_4$*.*

As in the sections before, we will prove this theorem with the help of the Atiyah-Hirzebruch spectral sequence. By Lemma 2.2.5 and Example 2.2.4, its second page

$$E^2_{p,q} = \ker \Big(\pi_p\colon H_p(B\mathrm{O}(2), \Omega_q^{\mathrm{Pin}^-}(\mathrm{pt})) \to H_p(\mathbb{R}P^\infty, \Omega_q^{\mathrm{Pin}^-}(\mathrm{pt})) \Big)$$

is partially described by Figure 3.5. Next, we list the differentials we are interested in.

Theorem 3.3.2.

1. $d^r_{p,q} = 0$ *for every* $r \geq 2$ *and* $p \leq 3$*.*
2. $d^2_{4,0} \neq 0$*.*
3. *Every differential with target* $E^r_{2,2}$ *vanishes for every* $r \geq 2$*.*
4. $d^2_{5,0} \neq 0$*.*

From this theorem we deduce the result for the third page of the spectral sequence listed in Figure 3.6.

Since $d^3_{5,0}$ has $E^3_{2,2}$ as target space, we conclude from the previous theorem that it must be zero. Therefore, we get $E^3_{p,q} = E^\infty_{p,q}$ if $p + q \leq 4$.

4	0	0	0	0	0	0
3	0	0	0	0	0	0
2	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	J	$\mathbb{Z}_2^2$
1	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
	0	1	2	3	4	5

Figure 3.5. The second page $E^2_{p,q}$ for $p \leq 5$ and $q \leq 4$ of the Atiyah-Hirzebruch spectral sequence for the kernel of $\Omega_*^{\mathrm{Pin}^-}(\pi)$. Here, $J = \mathbb{Z}_2 \oplus \mathbb{Z}_8$.

2	0	0	$\mathbb{Z}_2$	?	?	?
1	0	0	0	0	?	?
0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	0	1	2	3	4	5

Figure 3.6. The third page $E^3_{p,q}$ for $p \leq 5$ and $q \leq 4$ of the Atiyah-Hirzebruch spectral sequence for the kernel of $\Omega_*^{\mathrm{Pin}^-}(\pi)$.

Before we prove the theorem, let us consider the extension problem because it can be done rather easily. Forgetting the Pin^- structure gives a monomorphism

$$H_p\left(BO(2), \Omega_q^{\mathrm{Pin}^-}(\mathrm{pt})\right) \to H_p\left(BO(2), \Omega_q^{\mathrm{O}}(\mathrm{pt})\right)$$

if $q = 0$ or if $q = p = 2$. The case $q = 0$ follows from the fact that forgetting the Pin^- structure gives an isomorphism between the zero coefficient groups. For the case $p = q = 2$, observe that both homology groups are generated by $P(1,0) \otimes [\mathbb{R}P^2]$.

Thus, forgetting the Pin^- structure gives an monomorphism

$$\bigoplus_{k=p+q} {}^{\mathrm{Pin}^-}E^\infty_{p,q} \longrightarrow \bigoplus_{k=p+q} \ker {}^{\mathrm{O}}\pi^\infty_{p,q} \subseteq \bigoplus_{k=p+q} {}^{\mathrm{O}}E^\infty_{p,q}$$

for $k \leq 4$ because $\ker {}^{\mathrm{O}}\pi^\infty_{*,*} = \ker {}^{\mathrm{O}}\pi^2_{*,*}$. Therefore, it also induces a monomorphism between the kernels

$$\Omega_k^{\mathrm{Pin}^-}(BO(2)) \supseteq \ker {}^{\mathrm{Pin}^-}\pi_k \to \ker {}^{\mathrm{O}}\pi_k \subseteq \Omega_k^{\mathrm{O}}(BO(2)).$$

Since the right-hand-side is a $\mathbb{Z}_2$-vector space, the left-hand-side must be one, too. Consequently, $\ker \pi_4$ isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ instead of $\mathbb{Z}_4$, and the extension problem is solved.

Calculations of the differentials

We turn now to the verification of Theorem 3.3.2. This will be done by a sequence of lemmas.

Lemma 3.3.3. *For every $r \geq 2$ and $p \leq 3$, we have $d^r_{p,q} = 0$.*

Proof. The target space of $d^r_{p,q}$ is $E^r_{p-r,q+r-1} = 0$. □

Lemma 3.3.4. *Every differential with target space $E^r_{2,2}$ vanishes.*

Proof. The element $[P(1,0) \times \mathbb{R}P^2, \mathrm{pr}_2] \in {}^{\mathrm{Pin}^-}F_{2,2} \subseteq \Omega_4^{\mathrm{Pin}^-}(B\mathrm{O}(2))$ satisfies

$$\begin{aligned}\pi_4\big([P(1,0) \times \mathbb{R}P^2, \mathrm{pr}_2]\big) &= [P(1,0) \times \mathbb{R}P^2, \pi \circ \mathrm{pr}_2] \\ &= [P(1,0) \times \mathbb{R}P^2, \mathrm{const}] \\ &= 0 \in \Omega_4^{\mathrm{Pin}^-}(\mathrm{pt}),\end{aligned}$$

so it lies indeed in $\ker \pi_4$. Since $[P(1,0) \times \mathbb{R}P^2, \mathrm{pr}_2]$ lies in ${}^{\mathrm{O}}F_{2,2} \setminus {}^{\mathrm{O}}F_{1,3}$, $[P(1,0)\times\mathbb{R}P^2, \mathrm{pr}_2]$ lies in ${}^{\mathrm{Pin}^-}F_{2,2} \setminus {}^{\mathrm{Pin}^-}F_{1,3}$. From this we deduce $E^\infty_{2,2} \neq 0$. By Lemma 3.3.3, the differentials $d^r_{2,2}$ with $r \geq 2$ vanish. So, $E^r_{2,2}$ surjects onto $E^\infty_{2,2}$ for every $r \geq 2$. Now, if there would be a non trivial differential mapping into $E^r_{2,2}$, then $E^{r+1}_{2,2} = 0$, and therefore $E^\infty_{2,2} = 0$. But this contradicts the previous observation. □

Corollary 3.3.5. $d^2_{4,1} = 0$ *and* $d^3_{5,0} = 0$.

Lemma 3.3.6. $d^2_{4,0} \neq 0$.

Proof. Let $\{{}_{\mathbb{C}P^\infty}E^r_{p,q}\}$ be the Atiyah-Hirzebruch spectral sequence associated to $\Omega_*^{\mathrm{Pin}^-}(\mathbb{C}P^\infty)$. The inclusion

$$\mathbb{C}P^\infty \overset{\iota}{\hookrightarrow} B\mathrm{O}(2)$$

induces a map on spectral sequences

$${}_{\mathbb{C}P^\infty}E^r_{p,q} \xrightarrow{\iota^r_{p,q}} E^r_{p,q}$$

because $\pi \circ \iota = \mathrm{const}$. The map induces an inclusion if $r = 2$ and $q \in \{0,1\}$. Therefore, we conclude from the commutativity of the diagram

$$\begin{array}{ccc} {}_{\mathbb{C}P^\infty}E^2_{4,0} & \overset{\iota_{4,0}}{\hookrightarrow} & E^2_{4,0} \\ \Big\downarrow d^2_{4,0} \;\neq 0 & & \Big\downarrow d^2_{4,0} \\ \mathbb{Z}_2 \cong {}_{\mathbb{C}P^\infty}E^2_{2,1} & \underset{\iota_{2,1}}{\hookrightarrow} & E^2_{2,1} \cong \mathbb{Z}_2 \end{array}$$

that the right differential is also non-zero. □

Lemma 3.3.7. $d^2_{5,0} \neq 0$.

Proof. First, note that $E^\infty_{5,0} = E^2_{5,0}/\ker d^2_{5,0}$. Thus, if $d^2_{5,0} = 0$, the isomorphism $E^2_{5,0} \to \ker \pi^2_{5,0} \subseteq {}^{\mathrm{O}}E^2_{5,0}$ would induce an isomorphism $E^\infty_{5,0} \to \ker \pi^\infty_{5,0} = \ker \pi^2_{5,0} \subseteq {}^{\mathrm{O}}E^2_{5,0}$. So, there would be an element $[X,f]$ in $[P(2,1), \mathrm{incl}] + \mathrm{im}\,\big[\mathrm{incl}_*\colon \Omega_5(B\mathrm{O}(2)^{(4)}) \to \Omega_5^{\mathrm{O}}(B\mathrm{O}(2))\big]$ that can be represented by a singular Pin^- manifold and lies in the kernel of ${}^{\mathrm{O}}\pi_5$.

By applying $\mathrm{id} - (\sigma \circ \pi)_*$ to the generators of $\Omega_5^{\mathrm{O}}(B\mathrm{O}(2))$ described in Theorem 2.5.1, we conclude that $[X,f]$ is either represented by

$$[P(2,1), \mathrm{incl}] - [P(2,1), \sigma \circ \pi]$$

or

$$[P(2,1),\mathrm{incl}] - [P(2,1),\sigma\circ\pi] + \big[P(1,1)\times\mathbb{R}P^2,\mathrm{incl}\circ\mathrm{pr}_1\big] \\ - \big[P(1,1)\times\mathbb{R}P^2,\sigma\circ\pi\circ\mathrm{pr}_1\big]\,.$$

Under the identification $H^*(B\mathrm{O}(2);\mathbb{Z}_2)\cong\mathbb{Z}_2[x,y]$, we will verify that $[X,f]$ satisfies in both cases

$$\big((w_2-w_1^2)(X)\cup f^*(x\cdot y)\big)\cap[X]=1,$$

which gives a contradiction. To this end, observe that $(\sigma\circ\pi)^*(x\cdot y)=0$ because the homomorphism factors through $H^3(\mathbb{R}P^1)=0$. Since the generalised Stiefel-Whitney numbers are additive under disjoint unions, the stated equation will follow from the next two calculations. Identify $H^*(P(2,1);\mathbb{Z}_2)$ with $\mathbb{Z}_2[c,d]/\langle c^3,d^2\rangle$ and $H^*(P(1,1)\times\mathbb{R}P^2)$ with $\mathbb{Z}_2[c,d,a]/\langle c^2,d^2,a^3\rangle$ as described in Theorem 2.3.1. Then we get

$$\begin{aligned}&\big((w_2-w_1^2)(P(2,1))\cup\mathrm{incl}^*(x\cdot y)\big)\cap[P(2,1)]\\ &=\big((c-d^2)\cup c\cdot d\big)\cap[P(2,1)]\\ &=(c^2\cdot d)\cap([P(2,1)])=1,\end{aligned}$$

and

$$\begin{aligned}&\big((w_2-w_1^2)(P(1,1)\times\mathbb{R}P^2)\cup\mathrm{incl}^*(x\cdot y)\big)\cap[P(1,1)\times\mathbb{R}P^2]\\ &=\big((d\cdot a+a^2-d^2-a^2)\cup c\cdot d\big)\cap[P(1,1)\times\mathbb{R}P^2]\\ &=\big(a\cdot c\cdot d^2\big)\cap[P(1,1)\times\mathbb{R}P^2]=0\cap[P(1,1)\times\mathbb{R}P^2]=0.\end{aligned}$$

Even better, the proof shows that $[P(2,1),\mathrm{incl}]\in E^2_{5,0}$ does not lie in the kernel of $d^2_{5,0}$. □

Remark 3.3.8. The same argument shows that an arbitrary element $[X,f]$ in $[P(1,3),\mathrm{incl}]+\mathrm{im}\big[\mathrm{incl}_5\colon\Omega^{\mathrm{O}}_5(B\mathrm{O}(2)^{(4)})\to\Omega^{\mathrm{O}}_5(B\mathrm{O}(2))\big]$, that also lies in the kernel of ${}^{\mathrm{O}}\pi_5$, cannot be represented by a singular Pin^- manifold. Indeed,

$$\big((w_2-w_1^2)\cup\mathrm{incl}^*(x\cdot y)\big)\cap[P(1,3)]=\big((0+d^2)\cup c\cdot d\big)\cap[P(1,3)]=1,$$

and, therefore, any $[X,f]$ yields

$$\big((w_2-w_1^2)(X)\cup f^*(x\cdot y)\big)\cap[X]=1.$$

Consequently, $[P(1,3),\mathrm{incl}]\in E^2_{5,0}$ does not lie in $\ker d^2_{5,0}$. The same is true for $[P(2,1),\mathrm{incl}]\in E^2_{5,0}$. By a dimension count we conclude

$$\mathbb{Z}_2\cong\ker d^2_{5,0}=\mathrm{span}_{\mathbb{Z}_2}\{[P(1,3),\mathrm{incl}]+[P(2,1),\mathrm{incl}]\}\subseteq E^2_{5,0}.$$

Finally, we determine the generators of the kernel $\ker\pi_*$. It suffices to find enough generators in $\ker\pi_k$ that carry a Pin^- structure and do not represent the zero element in $\Omega^{\mathrm{O}}_k(B\mathrm{O}(2))$. Note that all singular Pin^- manifolds listed in Theorem 3.3.1 are non-zero in $\Omega^{\mathrm{O}}_*(B\mathrm{O}(2))$ by Theorem 2.5.1 and that they carry a Pin^- structure as one can easily calculate using Theorem 1.2.10 combined with Theorem 2.4.1. The two four-dimensional singular Pin^- manifolds obviously lie in the kernel of π_4 because $\Omega_4^{\mathrm{Pin}^-}(\mathbb{R}P^\infty) = 0$.

Since $P(1,0) = \mathbb{C}P^1 = S^2$ has D^3 as Pin^- boundary, we get

$$\begin{aligned}\pi_2([P(1,0), \mathrm{incl}]) &= [P(1,0), \pi \circ \mathrm{incl}] \\ &= [P(1,0), \mathrm{const}] \\ &= 0 \in \Omega_2^{\mathrm{Pin}^-}(\mathrm{pt}) \subseteq \Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty).\end{aligned}$$

To detect $[P(1,1), \mathrm{incl}]$ as generator for $\ker \pi_3$, we use the knowledge of $\Omega_3^{\mathrm{Pin}^-}$ from the previous section.

Lemma 3.3.9. $\pi_3([P(1,1), \mathrm{incl}]) = [P(1,1), \pi] = 0$.

Proof. Since the image of $\pi \circ \mathrm{incl}\colon P(1,1) \to \mathbb{R}P^\infty$ lies in $\mathbb{R}P^1$, the object of our interest must be zero or $[\mathbb{R}P^1 \times \mathbb{R}P^2, \mathrm{pr}_1]$. However, since $w_2(P(1,1)) = 0$, we get

$$(\pi^*(x) \cup w_2(P(1,1))) \cap [P(1,1)] = 0,$$

while

$$\big(\mathrm{pr}_1^*(x) \cup w_2(\mathbb{R}P^1 \times \mathbb{R}P^2)\big) \cap [\mathbb{R}P^1 \times \mathbb{R}P^2] = (a \cup b^2) \cap [\mathbb{R}P^1 \times \mathbb{R}P^2] = 1.$$

Here, x denotes the generator of $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$ and a, b the generators of $H^*(\mathbb{R}P^1 \times \mathbb{R}P^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[a,b]/\langle a^2, b^3 \rangle$. Therefore these two singular Pin^- manifolds cannot be bordant to each other and $[P(1,1), \pi]$ must be zero. □

Now, Theorem 3.2.1, Theorem 3.3.1, and Corollary 1.5.2 imply the solution of the motivating question of this thesis.

Theorem 3.3.10 (Main-Theorem)**.** *The first five* Pin^- *bordism groups of* $B\mathrm{O}(2)$ *are given by*

$$\begin{aligned}\Omega_0^{\mathrm{Pin}^-}(B\mathrm{O}(2)) &\cong \mathbb{Z}_2 \\ \Omega_1^{\mathrm{Pin}^-}(B\mathrm{O}(2)) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \Omega_2^{\mathrm{Pin}^-}(B\mathrm{O}(2)) &\cong \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\ \Omega_3^{\mathrm{Pin}^-}(B\mathrm{O}(2)) &\cong \mathbb{Z}_2^2 \oplus \mathbb{Z}_2 \\ \Omega_4^{\mathrm{Pin}^-}(B\mathrm{O}(2)) &\cong \mathbb{Z}_2^2.\end{aligned}$$

The generators are listed in the Theorems mentioned above.

Appendix A

Bundles and Classifying Spaces

We recall briefly basic facts about principal bundles and their classifying spaces. A nice introductions to principal G-bundles can be found in [tD08] and the appendix of [LM89].

A.1. Basic bundle theory

This section is a brief summary of Appendix C of [LM89]. No originality is claimed here.

Definition A.1.1. Let G be a topological group. A tuple $(P, B, \pi; R)$ consisting of a continuous map $\pi\colon P \to B$ between two topological spaces P and B and a continuous right action $r\colon P \times G \to P$ is called a *principal G-bundle* if the following conditions are satisfied:

1. The right action preserves the fibres: For all $p \in P$ and $g \in G$ we have $\pi(r(x, g)) = \pi(p)$.
2. There is an open cover $\{U_\alpha\}_{\alpha \in A}$ of B and equivariant homeomorphisms Φ_α satisfying

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\Phi_\alpha} & U_\alpha \times G \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U_\alpha. & \end{array}$$

The action on $U_\alpha \times G$ is given by $((x, g), h) \mapsto (x, gh)$. An open subset U_α as in 2. is called a *trivialisation domain.*

Definition A.1.2. A *morphism* between two principal G-bundles $P_j \xrightarrow{\pi_j} B_j$ is a continuous map $F\colon P \to P'$ that is equivariant with respect to the right actions of the bundles.

Two principal G-bundles P, P' over the same base space are called *equivalent* if there exists a morphism between them that induces the identity on the base space.

Definition A.1.3 (Cocycle)**.** Let B a topological space and $\{U_\alpha\}_{\alpha \in A}$ be an open cover of B. A *1-cocycle*, or simply *cocycle*, $\{g_{\alpha\beta}\}$ is a familiy of continuous functions $g_{\alpha\beta}\colon U_{\alpha\beta} = U_\alpha \cap U_\beta \to G$ satisfying the *cocycle condition*:

$$\begin{aligned} g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= 1 && \forall \alpha, \beta, \gamma \in A, \\ g_{\alpha\alpha} &= 1 && \forall \alpha \in A. \end{aligned}$$

Analogously, a *0-cocycle* $\{g_\alpha\}$ is a family of continuous function $g_\alpha\colon U_\alpha \to G$ such that $g_\alpha g_\beta^{-1} = 1$ on $U_{\alpha,\beta}$ for every $\alpha, \beta \in A$.

Definition A.1.4 (non-commutative Čech cohomology)**.** Two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ on an open cover $\mathcal{U}$ are *equivalent* if there is a family of continuous functions $g_\alpha \colon U_\alpha \to G$ such that

$$g'_{\alpha\beta} = g_\alpha^{-1} g_{\alpha\beta} g_\beta \text{for all} \alpha, \beta.$$

We define $H^1(\mathcal{U}, G)$ to be the set of equivalence classes of cocylces on $\mathcal{U}$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, that means for every $V \in \mathcal{V}$ there is an $U_V \in \mathcal{U}$ with $V \subseteq U_V$, then the restriction to the smaller subsets gives a well-defined map $r_{\mathcal{V}\mathcal{U}} \colon H^1(\mathcal{U}; G) \to H^1(\mathcal{V}; G)$ satisfying $r_{\mathcal{W}\mathcal{U}} = r_{\mathcal{W}\mathcal{V}} \circ r_{\mathcal{V}\mathcal{U}}$. Therefore we can define

$$\check{H}^1(B; G) := \lim_{\to} H^1(\mathcal{U}; G).$$

Analogously, we define $H^0(\mathcal{U}; G)$ to be set of all 0-cocylces and

$$\check{H}(B; G) = \lim_{\to} H^0(\mathcal{U}; G).$$

Note that for every open cover $\mathcal{U}$ the set $H^0(\mathcal{U}; G)$ is nothing but the set of continuous functions from B to G and so is $\check{H}^0(B; G)$, too. If G is abelian, then those groups agree with the usual Čech cohomolgy.

Theorem A.1.5. *There is a one-to-one correspondence between the isomorphism classes of principal G-bundles over the space B and elements in* $\check{H}^1(B; G)$. *This correspondence is natural with respect to continuous maps* $f \colon C \to B$.

Theorem A.1.6 (long exact Hirzebruch sequence)**.** *Let B be paracompact. A short exact sequence of topological groups*

$$1 \longrightarrow K \xrightarrow{i} G \xrightarrow{j} K \longrightarrow 1$$

with a local section over some neighbourhood of the unit 1_H *induces a long exact sequence*

$$0 \longrightarrow \check{H}^0(B; K) \xrightarrow{i_0} \check{H}^0(B; G) \xrightarrow{j_0} \check{H}^0(B; H) \xrightarrow{\delta_0} \dots$$

$$\dots \xrightarrow{\delta_0} \check{H}^1(B; K) \xrightarrow{i_1} \check{H}^1(B; G) \xrightarrow{j_1} \check{H}^1(B; H)$$

of pointed sets. This sequence is natural with respect to continuous maps. If K is central in G, then this sequence can be extended to

$$\dots \longrightarrow \check{H}^1(B, G) \xrightarrow{j_1} \check{H}^1(B; H) \xrightarrow{\delta_1} \check{H}^2(B; K).$$

Proof. We only give the definition of the connecting homomorphisms δ^0 and δ^1 and verify the exactness at $H^1(B; H)$ in the case that K is central in G.

Set $\delta^0(\{h_\alpha\}) = \{k_{\alpha\beta}\}$, where $k_{\alpha\beta}$ is defined as follows: Use the local section around the 1_H to lift $\{h_\alpha\}$ to a family of continuous function $\{g_\alpha\}$. Since, by definition, $h_\alpha \cdot h_\beta^{-1}|_{U_{\alpha\beta}} = 1$, we conclude $g_\alpha \cdot g_\beta^{-1} =: k_{\alpha\beta}$ takes values in $K = \ker j$. Obviously, $\{k_{\alpha\beta}\}$ is a 1−cocycle. Observe that different lifts give equivalent cocycles.

Analogously, we define

$$\delta^1(\{h_{\alpha\beta}\}) = \{g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}\} =: \{w_{\alpha\beta\gamma}\},$$

where $g_{\alpha\beta}$ is a lift of $h_{\alpha\beta}$. The map does not depend on the choice of the representative because of

$$\begin{aligned}\delta^1(\{h_\alpha^{-1}h_{\alpha\beta}h_\beta\}) &= \{g_\alpha^{-1}g_{\alpha\beta}g_\beta g_\beta^{-1}g_{\beta\gamma}g_\gamma g_\gamma^{-1}g_{\gamma\alpha}g_\alpha\}\\ &= \{g_\alpha^{-1}\underbrace{g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}}_{\in K}g_\alpha\} = \delta^1(\{h_{\alpha\beta}\}).\end{aligned}$$

Neither it depends on the choice of the lifts because they only differ by a coboundary as the following calculation shows:

$$\begin{aligned}\{g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}\} &= \{\bar{g}_{\alpha\beta}k_{\alpha\beta}\bar{g}_{\beta\gamma}k_{\beta\gamma}\bar{g}_{\gamma\alpha}k_{\gamma\alpha}\}\\ &= \{\bar{g}_{\alpha\beta}\bar{g}_{\beta\gamma}\bar{g}_{\gamma\alpha}k_{\alpha\beta}k_{\beta\gamma}k_{\gamma\alpha}\}.\end{aligned}$$

Now, $\delta^1 \circ j_1$ is the constant map with value $\{1\}$, so $\operatorname{im} j_1 \subseteq \ker \delta^1$. For the converse, let us assume that $\delta^1(\{h_\alpha\beta\})$ is cohomologous to $\{1\}$. This is equivalent to

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = k_{\alpha\beta}k_{\beta\gamma}k_{\gamma\alpha}.$$

Then $\{\bar{g}_{\alpha\beta}\} = \{g_{\alpha\beta}k_{\beta\alpha}\}$ is another lift of $\{h_{\alpha\beta}\}$ which happens to be a cocylce as the previous calculation shows. □

A.2. Numerable bundles

The source of this section is chapter 13 and chapter 14.3 of [tD08].

Definition A.2.1. Let B be a topological space. A covering $\{U_\alpha\}_{\alpha\in A}$ is called *numerable* if there is a locally finite partition of unity $(t_\alpha)_{\alpha\in A}$ with $\operatorname{supp}(t_\alpha) \subseteq U_\alpha$.

A principal G-bundle is called *numerable* if it has a numerable covering of trivialisation domains.

Lemma A.2.2. *Let $P \xrightarrow{\pi} B$ be a numerable bundle and $f\colon Y \to X$ be continuous. Then f^*P is a numerable bundle over Y.*

Theorem A.2.3. *Every open cover of a paracompact space is numerable.*

Theorem A.2.4. *Let $P \xrightarrow{\pi} B \times [0,1]$ be a numerable principal G-bundle. Then there exists an equivariant map $R\colon P \to P$ over $r\colon B \times [0,1] \to B \times [0,1]$, $(b,t) \mapsto (b,1)$ which is the identity on $P|_{B\times 1}$ and (R,r) is a pullback.*

Corollary A.2.5. *Let $P \xrightarrow{\pi} C$ be a numerable principal G-bundle. Furthermore, let $f,g\colon B \to C$ be two homotopic maps. Then f^*P is isomorphic to g^*P.*

A.3. Classifying spaces

The source for this section is chapter 14.4 of [tD08] unless stated otherwise.

Definition A.3.1 (classifying space)**.** Let G be a topological group and $\mathcal{B}(B;G)$ be the set of equivalence classes of numerable principal G-bundles over the topological space B. A topological space BG is called *classifying space* (of G) if there is a numerable bundle $EG \to BG$ such that for every topological space B the assignment

$$\begin{aligned}[B;BG] &\to \mathcal{B}(B;G)\\ [f] &\mapsto f^*EG\end{aligned}$$

is a bijection. The principal G-bundle EG is called *universal.*

Theorem A.3.2. *Any topological group has a classifying space BG and it is unique up to homotopy equivalence.*

Sketch of proof. Existence of the universal principal G-bundle follows from Milnor's join construction [Mil56]

$$EG = G * G * G * \dots,$$

and therefore $BG = EG/G$ exists. Uniqueness follows from general nonsense. □

Remark A.3.3. If G is a countable CW-group, i.e. a CW-complex with cellular group multiplication and cellular inversion, the join construction, endowed with the weak topology (quotient topology) instead of the Milnor topology, becomes a CW-complex. Milnor showed in his article [Mil56] that this join is still a universal principal G-bundle. The classifying space $BG = EG/G$ inherits a cell structure, too. This observation is quite useful, because the orthogonal groups $\mathrm{O}(n)$ are cellular groups. A cell structure on $\mathrm{SO}(n)$ making the multiplication and inversion cellular is described in Section 3.D in [Hat02]. It is easy to show that the conjugation with the reflection at the hyperplane orthogonal to $e_1 = (1, 0 \dots)$ is also cellular. Thus, $\mathrm{O}(n) = \mathrm{SO}(n) \rtimes \mathbb{Z}_2$ is also a cellular group, and so are $\mathrm{Pin}^{\pm}(n)$, $\mathrm{Spin}(n)$.

Proposition A.3.4. *The assignment $G \mapsto BG$ is functorial.*

Theorem A.3.5. *A principal G-bundle $P \to B$ is universal if and only if the total space P is contractible.*

Example A.3.6. A model for $B\mathrm{O}(1) = B\mathbb{Z}_2$ is given by $\mathbb{R}P^\infty$. Indeed, we have a two-sheeted covering

$$\mathbb{Z}_2 \longrightarrow S^\infty \longrightarrow \mathbb{R}P^\infty$$

and since S^∞ is contractible, conclude from Theorem A.3.5 that S^∞ is a universal principal $\mathbb{Z}_2$-bundle. Thus $\mathbb{R}P^\infty$ is a classifying space.

Analogously, the S^1 fibre bundle structure

$$S^1 \longrightarrow S^{2\infty+1} \longrightarrow \mathbb{C}P^\infty$$

identifies $\mathbb{C}P^\infty$ as $BS^1 = B\mathrm{U}(1)$. More generally, we have

$$B\mathrm{O}(n) = Gr_{\mathbb{R}}(n,\infty) = \lim_{N\to\infty} Gr_{\mathbb{R}}(n,N),$$
$$B\mathrm{U}(n) = Gr_{\mathbb{C}}(n,\infty) = \lim_{N\to\infty} Gr_{\mathbb{C}}(n,N).$$

Note that the Grassmannians are CW-complexes. A cell decomposition is given by the Schubert cells, see [Mil74, p.74ff] for more details.

Theorem A.3.7. *Let G be a topological group, $H \subseteq G$ a subgroup and $N \subseteq G$ a normal subgroup. Then there are models for the classifying space such that the maps*

$$G/H \hookrightarrow BH \xrightarrow{B\iota} BG,$$

$$BN \xhookrightarrow{B\iota} BG \xrightarrow{B\pi} B(G/N)$$

are fibre bundles.

We address the question for which groups G a classifying space BG is of the homotopy type of a CW-complex. As we have seen above, Milnor proved in [Mil56] that a classifying space possesses a CW-structure if the topological group G is a countable CW group, i.e., there is CW-structure on G with countably many cells such that the map $(x, y) \mapsto x^{-1}y$ is cellular. Although it seems likely that matrix groups have such a cell decomposition, a strict verification should be quite cumbersome.

Theorem A.3.8. *Let G be a matrix group or a covering of a matrix group with countable fibre. Then BG is of the homotopy type of a CW-complex.*

Proof. Firstly, we may assume that G is compact because it classifying space is homotopic to the classifying space of its maximal compact subgroup [tD08, p.348].

Let G be a compact matrix group. Since $\mathrm{O}(n)$ is the maximal subgroup of $\mathrm{GL}(n)$, by definition, $G \subseteq \mathrm{O}(n)$ must be a subgroup. So from Theorem A.3.7 we have a fibre bundle structure

$$\mathrm{O}(n)/G \longrightarrow BG \xrightarrow{B\iota} B\mathrm{O}(n) = Gr_{\mathbb{R}}(n,\infty)\ .$$

Since a fibre bundle over a paracompact space is a fibration, see [tD08, Thm.13.4.3], $\mathrm{O}(n)/G$ as a manifold is of the homotopy type of a CW-complex, see [FP90, Cor.5.2.4], and $Gr_{\mathbb{R}}(n,\infty)$ is a path connected CW-complex, we conclude from Theorem 5.4.2 in [FP90] that BG is of the homotopy type of a CW-complex.

A covering $p\colon G \to H \subseteq \mathrm{O}(n)$ gives a short exact sequence

$$1 \longrightarrow \ker p \hookrightarrow G \xrightarrow{p} H \longrightarrow 1$$

with discrete kernel. Clearly, every countable discrete group has a countable CW-structure consisting of 0-cells only. Therefore, any map is cellular and, by Theorem 5.2 of [Mil56], $B(\ker p)$ is homotopy equivalent to a

CW-complex. By the previous discussion, BH is homotopy equivalent to a CW-complex. Pull back this fibre bundle with the homotopy equivalence to obtain a $B\ker p$-fibre bundle over a CW-complex which is homotopy equivalent to BG. Therefore, we might assume that BH is a CW-complex in the first place and since any CW-complex is paracompact, we see that $Bp\colon BG \to BH$ is a fibration having CW-complexes as base and fibres. Now, we apply Theorem 5.4.2 in [FP90] to conclude that BG is homotopy equivalent to a CW-complex. □

A.4. Reductions of principal bundles

Definition A.4.1. Let $\lambda\colon H \to G$ be a homomorphism of topological groups and $P \to B$ be a principal G-bundle. A *(H,λ)-reduction of P*, or simply *H-reduction*, if it is clear which homomorphism is used, is a principal H-bundle $Q \to B$, together with an equivariant map $\rho\colon Q \to P$ covering the identity on B.

Two (H,λ)-reductions $\rho_j\colon Q_j \to P$ are called *equivalent* if there is an equivariant map θ such that the following diagram

$$\begin{array}{ccc} Q_1 & \xrightarrow{\theta} & Q_2 \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\ P & \xrightarrow{\mathrm{id}} & P \end{array}$$

commutes.

Theorem A.4.2. *Let $P \to B$ be a principal G-bundle. Then the following assertions are equivalent:*

1. *There exists a (H,λ)-reduction for P.*
2. *There is a principal H-bundle Q such that $P \cong Q \times_{(H,\lambda)} G$.*
3. *There is a H-valued cocylce $\{h_\alpha\beta\}$ such that P is represented by the cocycle $\{\lambda(h_{\alpha\beta})\}$.*
4. *In the case that λ is an inclusion of a subgroup there exists a continuous equivariant map $f\colon P \to G/H$ such that $Q = f^{-1}(g_0/H)$, for some $g_0 \in G$.*

If P is additionally numerable, then the assertions are equivalent to:

5. *The diagram*

$$\begin{array}{ccc} B & \xrightarrow{f_P} & BG \\ & {\scriptstyle f_Q}\searrow & \uparrow{\scriptstyle B\lambda} \\ & & BH \end{array}$$

commutes upto homotopy.

Proof. For the equivalence of the first four statements see [Bau14]. Observe that condition 5 is equivalent to condition 2. Indeed, if 4 holds, then

$$\begin{aligned} P \cong f_P^* EG = (B\lambda \circ f_Q)^* EG = f_Q^* \left(B\lambda^* EG\right) \\ \cong f_Q^* \left(EH \times_{(H,\lambda)} G\right) = f_Q^* EH \times_{(H,\lambda)} G \\ \cong Q \times_{H,\lambda} G. \end{aligned}$$

Conversely, if $P \cong Q \times_{H,\lambda} G = f_Q^* EH \times_{(H,\lambda)} G$, then the pullback of $B\lambda \circ f_Q$ and f_P give isomorphic principal bundles, and f_P and $f_Q \circ B\lambda$ must be therefore homotopic. □

Appendix B

Bordism theories

Definition B.0.1 (stable structures)**.** Let $\mathbf{G} = \{G_n, \varphi_n, \rho_n\}$ be a sequence of topological groups together with continuous homomorphisms $\varphi_n \colon G_n \to G_{n+1}$ and $\rho_n \colon G_n \to \mathrm{O}(n)$ satisfying

$$\begin{array}{ccc} G_n & \xrightarrow{\varphi_n} & G_{n+1} \\ \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho_{n+1}} \\ \mathrm{O}(n) & \hookrightarrow & \mathrm{O}(n+1). \end{array}$$

The last morphism is just given by the canonical inclusion

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$

We call this sequence a *stable* $\mathbf{G}$ *structure.*

A stable $\mathbf{G}$ structure on a principal $\mathrm{O}(n)$-bundle P is a sequence of principal G_n-bundles over the same base space M and equivariant maps over id_M making the following diagram

$$\begin{array}{ccccccccc} Q_n & \to & \dots & \to & Q_{n+k} & \to & Q_{n+k+1} & \to & \dots \\ \downarrow & & & & \downarrow & & \downarrow & & \\ P & \hookrightarrow & \dots & \hookrightarrow & P \times_{\mathrm{O}(n)} \mathrm{O}(n+k) & \hookrightarrow & P \times_{\mathrm{O}(n)} \mathrm{O}(n+k+1) & \hookrightarrow & \dots \end{array}$$

commutative. Note that all maps in the diagram are part of the structure. Equivalently, one can define stable $\mathbf{G}$ structures using classifying spaces as it is done in [Sto15] in an even more general setting.

Two stable structures $\mathbf{G}, \mathbf{G}'$ are *equivalent* if there is a $m \in \mathbb{N}$ and a sequence of equivariant maps $\theta_n \colon Q_n \to Q'_n$ starting at m that commutes with all maps in the definition of a stable structure and covers the identity of the standard orthonormal stable structure $\dots P \times_{\mathrm{O}(m)} \mathrm{O}(m+n-1) \hookrightarrow P \times_{\mathrm{O}(m)} \mathrm{O}(m+n) \dots$.

Example B.0.2. In Lemma 1.2.15, we have shown that possessing a $\mathrm{Pin}^{\pm}$ structure is a stable property. The sequence $\mathrm{Pin}^{\pm}$ is given by

$$\begin{array}{ccc} \mathrm{Pin}^{\pm}(n) & \longrightarrow & \mathrm{Pin}^{\pm}(n+1) \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda} \\ \mathrm{O}(n) & \hookrightarrow & \mathrm{O}(n+1). \end{array}$$

If $\rho\colon Q \to P$ is a $\mathrm{Pin}^\pm$ on P, then the corresponding stable $\mathrm{Pin}^\pm$ structure is given by

$$\begin{array}{ccccccc} \dots \longrightarrow & Q \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+k) & \longrightarrow & Q \times_{\mathrm{Pin}^\pm(n)} \mathrm{Pin}^\pm(n+k+1) & \longrightarrow \dots \\ & \Big\downarrow{\scriptstyle \rho\times\lambda} & & \Big\downarrow{\scriptstyle \rho\times\lambda} & \\ \dots \longrightarrow & P \times_{\mathrm{O}(n)} \mathrm{O}(n+k) & \hookrightarrow & P \times_{\mathrm{O}(n)} \mathrm{O}(n+k+1) & \hookrightarrow \dots . \end{array}$$

The next definition is taken from [DK01, p.216]. However, nice references are also [Swi02, p.226] and [Sto15].

Definition B.0.3 (G bordism groups)**.** The *n-th* **G** *bordism group* of a topological space X is the set of **G** bordism classes of closed singular manifolds $f\colon M \to X$ of dimension n that possesses a stable **G** structure on their normal bundles. The addition is given by disjoint union.

More generally, the n-th **G** bordism group $\Omega_n^g(X,A)$ of a pair (X,A) is the equivalence classes of singular n-dimensional manifolds $f\colon (M,\partial M) \to (X,A)$ with a stable **G** structure on its normal bundle. Two singular manifolds (M_i, f_i) are considered to be equivalent if there is a singular bordism (U,f) with a stable **G** structure on its normal bundle such that the closed singular manifold $(M_0 \cup_{\partial M_0} \overline{U} \cup_{\partial M_1} M_1, f_0 \cup f \cup f_1)$, obtained by proper boundary identifications, possesses a stable **G** structure on its normal bundle and is **G** bordant to the empty set. Again, the addition is given by disjoint union.

Remark B.0.4. Although we have used embeddings to define a bordism class, this class is independent of the used embedding. Indeed, let ι_0, ι_1 be two embeddings of a compact manifold M into some sufficiently high-dimensional euclidean space with non-intersecting images. We denote by $(M, \mathbf{G}_{\iota_0})$ and $(M, \mathbf{G}_{\iota_1})$ the manifold M with a chosen **G** structure on the bundle induced by the embeddings. Two embeddings are ambient-isotopic if the target space is sufficiently high-dimensional (cf. [Swi02, Theorem 12.14]) by an ambient isotopy H. This isotopy gives an isomorphism of normal bundles and we can use this isomorphism to pull back the chosen $\mathbf{G}_{\iota_1}$ structure to the stable normal bundle $\nu(\iota_0)$. If this structure is equivalent to $\mathbf{G}_\iota$ the bordism classes generated by $M, \mathbf{G}_{\iota_0}$ and $M, \mathbf{G}_{\iota_1}$ will be equal. For a more detailed explanation, the reader is reffered to chapter II of [Sto15].

The next theorem relates the very geometric definition to homotopy theory. Excellent references for the proof are [Swi02, p.230], [DK01, p.221ff], [Sto15]. A nice treatment of the special cases $\mathbf{G} = \mathbf{O}, \mathbf{SO}$ can be found in [Hir12] and [tD08].

Theorem B.0.5 (Pontrjagin-Thom)**.** *The Pontrjagin-Thom construction gives an isomorphism*

$$\Omega_n^G(X,A) \cong \lim_{k\to\infty} \pi_k(MG_k \wedge X_+/A_+)$$

Several bordism theories have been studied quite successfully. However, in this appendix we summarise the needed information for ordinary bordism ($\mathbf{G} = \mathbf{O}$), oriented bordism ($\mathbf{G} = \mathbf{SO}$), and Spin bordism $\mathbf{G} = \mathbf{Spin}$.

B.1. Unoriented bordism

Theorem B.1.1. [Dol56, Satz 3] *The set* $\Omega^{\mathrm{O}}_*(\mathrm{pt}) = \bigoplus_{n\geq 0}\Omega^{\mathrm{O}}_n(\mathrm{pt})$ *together with disjoint union and Cartesian product forms a graded commutative algebra over the field* $\mathbb{Z}_2$.

As an algebra,

$$\Omega^{\mathrm{O}}_* \cong \mathbb{Z}_2[\{x_j \,|\, j \neq 2^k - 1 \text{ for } k \in \mathbb{N}\}],$$

where $x_j = [\mathbb{R}P^j]$, *if* j *is even, and* $x_j = [P(2^r-1, s2^r)]$ *if* j *is odd.*

Example B.1.2.

$n =$	0	1	2	3	4	5	6	7
$\Omega^{\mathrm{O}}_n(\mathrm{pt}) \cong$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$

Theorem B.1.3. [Sto15, p.108] *The generalised Hurewicz map*

$$\Omega^{\mathrm{O}}_n(X,A) \to H_n(X,A;\mathbb{Z}_2),$$
$$[M, \partial M, F] \mapsto F_*[M, \partial M],$$

which sends a bordism class of a singular manifold to the image of its fundamental class, is an epimorphism.

Theorem B.1.4. [Sto15, p.107ff] *For every CW-pair* (X, A), *the unoriented bordism group* $\Omega^{\mathrm{O}}_*(X,A)$ *is -a free* $\Omega^{\mathrm{O}}_*(\mathrm{pt})$ *module, and there is a isomorphism of graded* $\Omega^{\mathrm{O}}_*(\mathrm{pt})$*-modules*

$$\Omega^{\mathrm{O}}_*(X,A) \cong H_*(X,A) \otimes_{\mathbb{Z}_2} \Omega^{\mathrm{O}}_*(\mathrm{pt}).$$

Moreover, two elements $[M_i, \partial M_i, f_i]$ *in* $\Omega^{\mathrm{O}}_*(X,A)$ *agree if and only if all generalised Stiefel-Whitney numbers*

$$(w_{i_1} \cup \ldots w_{i_k} \cup f^*(x)) \cap [M_i, \partial M_i]$$

agree. Here, $x \in H^{i_{k+1}}(X,A;\mathbb{Z}_2)$ *is chosen such that* $i_1 + \cdots + i_{k+1} = \dim M$.

B.2. Oriented bordism

Theorem B.2.1. *The set* $\Omega^{\mathrm{SO}}_*(\mathrm{pt})$ *together with disjoint union and Cartesian product forms a graded commutative ring.*

Theorem B.2.2. [tD08, Theorem 21.4.2] *The generalised Hurewicz map extends to an isomorphism of rings*

$$\Omega^{\mathrm{SO}}_*(\mathrm{pt}) \otimes \mathbb{Q} \cong H_*(B\mathrm{SO};\mathbb{Q}) \cong \mathbb{Q}[x_{4j}].$$

The generators are given $[\mathbb{C}P^{2j}]$. *The product on the left-hand side is induced by the Cartesian products of the representatives.*

Example B.2.3. [Mil74, p.203]

$n =$	0	1	2	3	4	5	6	7
$\Omega^{\mathrm{SO}}_n(\mathrm{pt}) \cong$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0

Theorem B.2.4. [Wal60] *Oriented bordism is completely characterised by the Pontrjagin number and the Stiefel-Whitney numbers. More precisely,* $[M] = 0 \in \Omega_n^{\mathrm{SO}}$ *if and only if all Stiefel-Whitney numbers and all Pontrjagin numbers vanish.*

Theorem B.2.5. [tD08, p.527] *The generalised Hurewicz map*

$$\begin{aligned} \Omega_n^{\mathrm{SO}}(X, A) &\to H_n(X, A; \mathbb{Z}), \\ [M, \partial M, F] &\mapsto F_*[M, \partial M], \end{aligned}$$

which sends a bordism class of a singular manifold to the image of its fundamental class is a well-defined homomorphism; it is neither injective nor surjective.

B.3. Spin bordism

Theorem B.3.1. *The* Spin *bordism coefficient group* $\Omega_*^{\mathrm{Spin}}(\mathrm{pt})$ *together with disjoint union and Cartesian product forms a ring.*

Example B.3.2. [Mil63]

$n =$	0	1	2	3	4	5	6	7
$\Omega_n^{\mathrm{Spin}}(\mathrm{pt}) \cong$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

The group $\Omega_1^{\mathrm{Spin}}(\mathrm{pt})$ is generated by $[S^1_{\mathrm{Lie}}]$, the circle with the 'bad' Spin structure, $\Omega_2^{\mathrm{Spin}}(\mathrm{pt})$ is generated by $S^1_{\mathrm{Lie}} \times S^1_{\mathrm{Lie}}$, and $\Omega_4^{\mathrm{Spin}}(\mathrm{pt})$ is generated by the Kummer surface.

Theorem B.3.3. [ABP67] *A class* $[M] \in \Omega_*^{\mathrm{Spin}}(\mathrm{pt})$ *is uniquely determined by its Stiefel-Whitney numbers and its* KO*-Pontrjagin numbers. More precisely,* $[M] = 0$ *if and only if all Stiefel-Whitney numbers and all* KO*-Pontrjagin classes vanish.*

Appendix C

Spectral Sequences

Definition C.0.1 (spectral sequence)**.** [McC01, p.28ff] Let R be a commutative ring with unit. A *spectral sequence of homological type* is a sequence of bigraded chain complexes $(E^r_{*,*}, d^r)$ of R-modules, where the differentials are of bidegree $(-r, r-1)$, and for every $p, q \in \mathbb{Z}$ and $r \in \mathbb{N}$ the groups $E^{r+1}_{p,q}$ and $H_{p,q}(E^r_{*,*}, d^r)$ are isomorphic.

Definition C.0.2. A spectral sequence is a *first quadrant* spectral sequence if, for every $r \in N$, we have $E^r_{p,q} = 0$ as long as $p < 0$ or $q < 0$.

Definition C.0.3 (morphism of spectral sequence)**.** [McC01, p.65]
A *morphism of spectral sequences* is a sequence of bigraded chain maps $f^r \colon E^r_{*,*} \to E^r_{*,*}$ of bidegree $(0,0)$ such that for every $p, q \in \mathbb{Z}$ and $r \in \mathbb{N}$, the identity $H_{p,q}(f^r) = f^{r+1}_{p,q}$ holds.

Definition C.0.4 (filtration)**.** [DK01, Definition 9.2] A *filtration* of a graded module M_* is an increasing sequence $(F_q)_{q\in\mathbb{Z}}$ of submodules of M_*. It is said to be *grading-preserving*, if for each $q \in \mathbb{Z}$ the intersections $F_{p,q} := F_p \cap M_{p+q}$ form a filtration of M_{p+q}. A filtration is *convergent* if the intersection of all F_p is 0 and the union is the whole module M_*.

Definition C.0.5 (convergence of spectral sequence)**.** [DK01, Definition 9.5]
Let M_* be a graded module. A spectral sequence *converges to* M_* if the following conditions are satisfied:

1. For every $p, q \in \mathbb{Z}$, there exists an r_0 such that $d^r_{p,q} = 0$ for every $r \geq r_0$. (This implies that $E^r_{p,q}$ surjects onto $E^{r+1}_{p,q}$ for every $r \geq r_0$.)
2. The graded module M_* possesses a convergent, grading-preserving filtration which satisfies $F_{p,q}/F_{p-1,q+1} \cong \varinjlim_r E^r_{p,q}$.

One often writes $E^2_{p,q} \Rightarrow M_{p+q}$ to denote convergence.

Remark C.0.6. First quadrant spectral sequences always satisfy the first condition of the convergence requirements. Indeed, for every $r > \max\{p,q\}$, we have $d^r_{p,q} = 0$ because its target space is 0, and $d^r_{p+r,q-r+1} = 0$ because its domain is 0. This implies $E^r_{p,q} \cong E^{r+1}_{p,q}$ for every $r > \max\{p,q\}$.

Definition C.0.7. [McC01, p.17ff] Let Γ^* be a graded algebra over R. We say that Γ^* *acts on* a spectral sequence if the following condition are satisfied:

1. Γ^* acts on the graded module $E^r_{*,*}$.
2. Every differential is Γ^*-linear.
3. The Γ^* action on $E^{r+1}_{*,*}$ is induced through homology from the action of Γ^* on E^r_{**}.

We call such a spectral sequence a *spectral sequence of* Γ^**-modules*.

Let M_* be a Γ^*-module. We say that a spectral sequence of Γ^*-modules converges to M_* *as* Γ^**-modules* if it converges to M_* and, additionally, the chosen filtration is Γ^*-invariant and the induced action on every $F_{p,q}/F_{p-1,q+1}$ agrees with the induced action on $\underrightarrow{\operatorname{colim}}_r\, E^r_{p,q}$.

We will construct a spectral sequence approximating a homology theory of a CW-complex. There are several approaches, we will follow the version covered in [Swi02], where all the proofs can be found if they are not given here. In what follows, X denotes a filtered topological space with filtration $(X^q)_{q\in\mathbb{Z}}$ with $X^q = \emptyset$ if $q < 0$ and h_* a generalised unreduced homology theory satisfying the axiom of disjoint union, like bordism theories.

Definition C.0.8. Let i and j be inclusions and Δ be the boundary operator for the homology theory h_*. We define

$$\begin{aligned}
Z^r_{p,q} &:= \operatorname{im}[j_*\colon h_{p+q}(X^p, X^{p-r}) \to h_{p+q}(X^p, X^{p-1})],\\
B^r_{p,q} &:= \operatorname{im}[\Delta\colon h_{p+q+1}(X^{p+r-1}, X^p) \to h_{p+q}(X^p, X^{p-1})],\\
Z^\infty_{p,q} &:= \operatorname{im}[j_*\colon h_{p+q}(X^p) \to h_{p+q}(X^p, X^{p-1})],\\
B^\infty_{p,q} &:= \operatorname{im}[\Delta\colon h_{p+q+1}(X, X^p) \to h_{p+q}(X^p, X^{p-1})],\\
F_{p,q} &:= \operatorname{im}[i_*\colon h_{p+q}(X^p) \to h_{p+q}(X)].
\end{aligned}$$

Proposition C.0.9. *For fixed p and q, the groups $B^r_{p,q}$, $Z^r_{p,q}$, $B^\infty_{p,q}$, and $Z^\infty_{p,q}$ are subgroups of $h_{p+q}(X^p, X^{p-1})$ and satisfy the following inclusion relations:*

$$\begin{aligned}
0 = B^1_{p,q} \subseteq \cdots \subseteq B^r_{p,q} \subseteq B^{r+1}_{p,q} \subseteq \cdots \subseteq B^\infty_{p,q} \subseteq Z^\infty_{p,q} \subseteq \cdots \subseteq Z^{r+1}_{p,q} \subseteq Z^r_{p,q} \subseteq \dots\\
\dots \subseteq Z^1_{p,q} = h_{p+q}(X^p, X^{p-1}).
\end{aligned}$$

Definition C.0.10. For $r \in \mathbb{N} \cup \{\infty\}$ define $E^r_{p,q} := Z^r_{p,q}/B^r_{p,q}$.

Lemma C.0.11. *The morphisms $j_*\colon h_{p+q}(X^p, X^{p-r}) \to h_{p+q}(X^p, X^{p-1})$ and $\Delta\colon h_{p+q}(X^p, X^{p-r}) \to h_{p+q-1}(X^{p-r}, X^{p-r-1})$ induce an isomorphism*

$$Z^r_{p,q}/Z^{r+1}_{p,q} \cong B^{r+1}_{p-r,q+r-1}/B^r_{p-r,q+r-1}$$

via $\Delta \circ j_*^{-1}$.

Definition C.0.12. We define the differentials $d^r_{p,q}$ as the composition in the diagram

$$\begin{array}{ccccccc}
Z^r_{p,q}/B^r_{p,q} & \twoheadrightarrow & Z^r_{p,q}/Z^{r+1}_{p,q} & \xrightarrow{\cong} & B^{r+1}_{p-r,q+r-1}/B^r_{p-r,q+r+1} & \hookrightarrow & Z^r_{p-r,q+r-1}/B^r_{p-r,q+r-1}\\
\| & & & & & & \|\\
E^r_{p,q} & & & \xrightarrow{d^r_{p,q}} & & & E^r_{p-r,q+r-1}.
\end{array}$$

Proposition C.0.13.

1. $\ker d^r_{p,q} = Z^{r+1}_{p,q}/B^r_{p,q}$ *and* $\operatorname{im} d^r_{p,q} = B^{r+1}_{p-r,q+r-1}/B^r_{p-r,q+r-1}$.
2. $(E^r_{*,*}, d^r)$ *is a bigraded chain complex for every $r \in \mathbb{N}$.*
3. $E^{r+1}_{*,*} \cong H(E^r_{*,*}, d^r)$.

Proposition C.0.14. *The subgroups $F_{p,q}$ form a convergent filtration for $h_{p+q}(X)$. Moreover, there is a natural isomorphism*

$$F_{p,q}/F_{p-1.q+1} \cong E^{\infty}_{p,q}$$

induced by $i_* \circ j_*^{-1}$, where $i_* \colon h_{p+q}(X^p) \to h_{p+q}(X)$ and $j_* \colon h_{p+q}(X^p) \to h_{p+q}(X^p, X^{p-1})$.

Proposition C.0.15. *$Z^r_{p,q} = Z^{\infty}_{p,q}$ for $r > p$, so we have an epimorphism $E^r_{p,q} \to E^{r+1}_{p,q}$. Furthermore, $B^{\infty} = \bigcup_{r\geq 1} B^r_{p,q}$ and thus $E^{\infty}_{p,q} \cong \underrightarrow{\operatorname{colim}}_r E^r_{p,q}$.*

Now we restrict our consideration to the case where X is a CW-complex and the filtration is the skeleton filtration.

Proposition C.0.16. *There is a natural isomorphism $E^1_{p,q} \cong \mathcal{C}^{cell}_p(X) \otimes h_q(\mathrm{pt})$ and under this isomorphism the differential $d^1_{p,q}$ can be described in terms of the cellular differential. More precisely, the diagram*

$$\begin{array}{ccc} E^1_{p,q} & \xrightarrow{d^1_{p,q}} & E^1_{p-1,q+1} \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{C}^{cell}_p(X) \otimes h_q(\mathrm{pt}) & \xrightarrow{\partial_p \otimes \mathrm{id}} & \mathcal{C}^p_{cell}(X) \otimes h_q(\mathrm{pt}) \end{array}$$

commutes.

The next theorem summarises the achievements of the previous propositions.

Theorem C.0.17 (Atiyah-Hirzebruch spectral sequence)**.** *We denote with X a CW-complex and h_* a generalised unreduced homology theory satisfying the axiom of disjoint union. Then there is a spectral sequence $\{E^r_{p,q}, d^r\}$ of homological type with $(E^1_{p,q}, d^1)$ as described in Proposition C.0.16, $E^2_{p,q} = H_p(X, h_q(\mathrm{pt}))$, and converging to $h_*(X)$.*

In the spirit of the leitfaden of Switzer's book, enrichment of structure, we are going to exploit further properties of this sequence.

Theorem C.0.18 (functionriality)**.** *A continuous map $f \colon X \to Y$ between two CW-complexes induces a map of spectral sequences $f^r_{p,q} \colon {}_XE^r_{p,q} \to {}_YE^r_{p,q}$, which approximates $h_*(f) \colon h_*(X) \to h_*(Y)$.*

Under the correspondence $E^1_{p,q} \cong \mathcal{C}^{cell}_p(X) \otimes h_q(\mathrm{pt})$ the maps $f^1_{p,q}$ correspond to $\mathcal{C}^{cell}_p(f) \otimes \mathrm{id} \colon \mathcal{C}^{cell}_p(X) \otimes h_q(\mathrm{pt}) \to \mathcal{C}^{cell}_p(Y) \otimes h_q(\mathrm{pt})$.

Proof. We may assume that f preserves the filtration because the cellular approximation theorem guarantees we can homotopy f into a cellular map if needed. Let ${}_XZ^r_{p,q}, {}_XB^r_{p,q}, \ldots$ the groups defined in Definition C.0.8 for the space X and ${}_YZ^r_{p,q}, {}_YB^r_{p,q}, \ldots$ the groups for Y. Since f is cellular, the diagrams

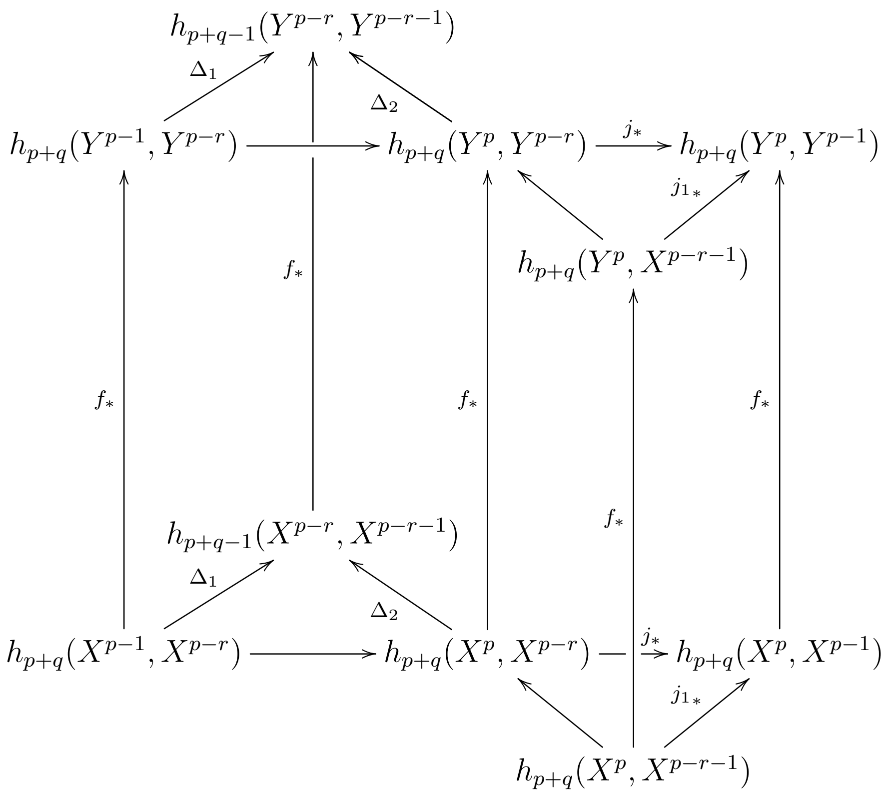
$$\begin{array}{ccc} h_*(X^\alpha, X^\beta) & \xrightarrow{j_*} & h_*(X^p, X^q) \\ \downarrow f_* & & \downarrow f_* \\ h_*(Y^\alpha, Y^\beta) & \xrightarrow{j_*} & h_*(Y^p, Y^q) \end{array} \qquad \begin{array}{ccc} h_*(X^\alpha, X^\beta) & \xrightarrow{\Delta} & h_*(X^p, X^q) \\ \downarrow f_* & & \downarrow f_* \\ h_*(Y^\alpha, Y^\beta) & \xrightarrow{\Delta} & h_*(Y^p, Y^q) \end{array}$$

commutes. This implies that the chain of inclusions given in Proposition C.0.9 is natural with respect to cellular maps. So, cellular maps define maps $f^r_{p,q}$ between ${}_XZ^r_{p,q}, {}_XB^r_{p,q}, \dots$ and ${}_YZ^r_{p,q}, {}_YB^r_{p,q}, \dots$. Thus, they induce well defined maps $f^r_{p,q}\colon {}_XE^r_{p,q} \to {}_YE^r_{p,q}$. The next step is to show that $\{f^r_{p,q}\}\colon \{{}_XE^r_{p,q}, d^r\} \to \{{}_YE^r_{p,q}, d^r\}$ is a map of spectral sequence. So, we need to verify that they are chain maps and that $H(f^r_{p,q}) = f^{r+1}_{p,q}$.

Recall that $d^r_{p,q}$ is the composition of the three homomorphisms

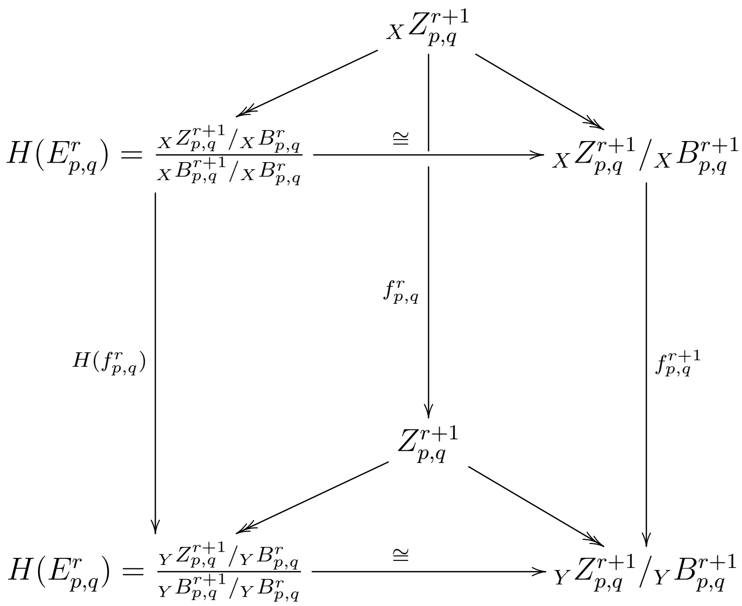
$$Z^r_{p,q}/B^r_{p,q} \twoheadrightarrow Z^r_{p,q}/Z^{r+1}_{p,q} \xrightarrow{\cong} B^{r+1}_{p-r,q+r-1}/B^r_{p-r,q+r+1} \hookrightarrow Z^r_{p-r,q+r-1}/B^r_{p-r,q+r-1}.$$

The first and the third morphism obviously commute with $f^r_{p,q}$. A diagram chase of

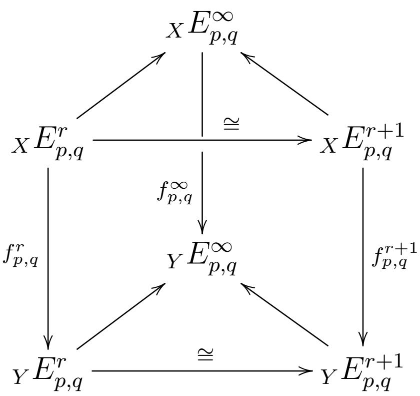


shows that the second morphism, induced by $\Delta_2 \circ j_*^{-1}$, commutes with $f^r_{p,q}$, too. So $\{f^r_{p,q}\}$ is a chain map for every fixed r.

The commutativity of



shows $H(f^r_{p,q}) = f^{r+1}_{p,q}$. Since $Z^r_{p,q} = Z^\infty_{p,q}$ for large r, we conclude from the previous discussion that $\{f^r_{p,q}\}$ is a morphism of directed systems commuting with $f^\infty_{p,q}$ because the following diagram

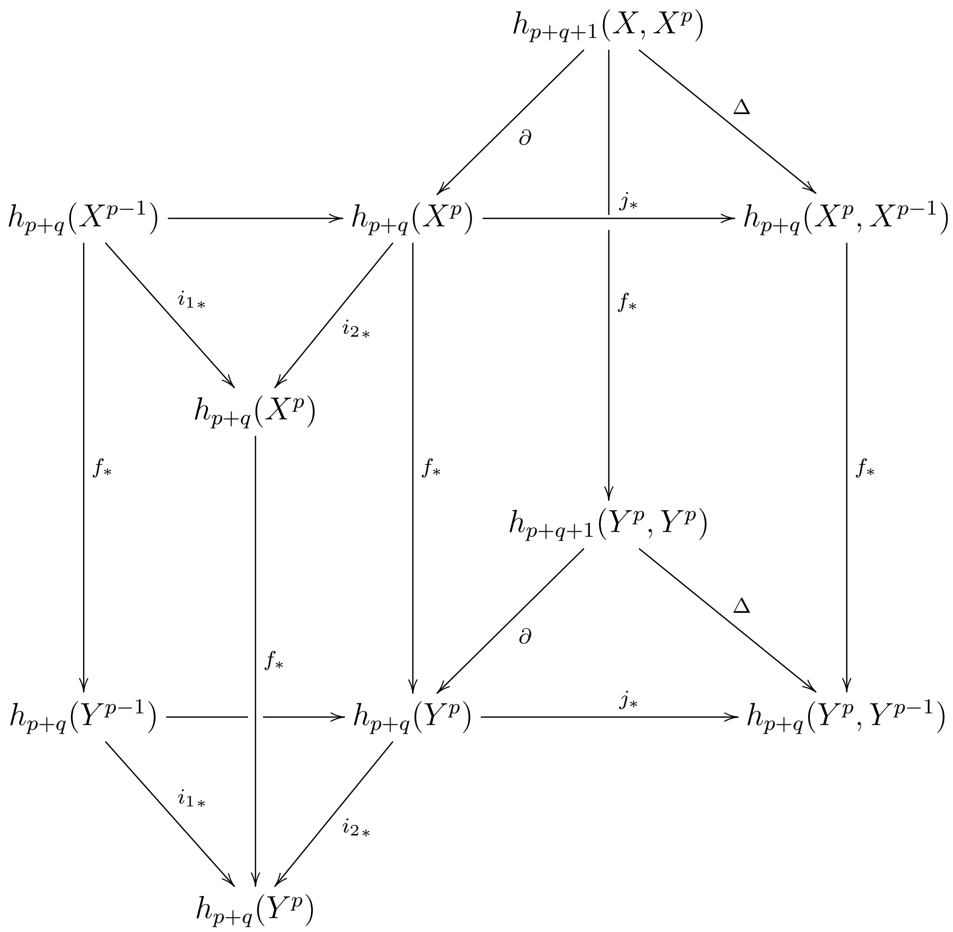


commutes. Consequently, we have

$$\begin{array}{ccc} \operatorname{colim}_r {}_XE^r_{p,q} & \xrightarrow{\cong} & {}_XE^\infty_{p,q} \\ \downarrow{\scriptstyle \operatorname{colim}\{f^r_{p,q}\}} & & \downarrow{\scriptstyle f^\infty_{p,q}} \\ \operatorname{colim}_r {}_YE^r_{p,q} & \xrightarrow{\cong} & {}_YE^\infty_{p,q}. \end{array}$$

The last condition we have to verify is whether the isomorphism between $F_{p,q}/F_{p-1,q+1}$ and $E^\infty_{p,q}$ is natural with respect to the morphisms induced by

f. To this end, consider the commutative diagram



and recall that the isomorphism between $F_{p,q}/F_{p-1,q+1}$ and $E^\infty_{p,q}$ is induced by $i_{2*} \circ j_*^{-1}$. A diagram chase shows the desired result.

The second part of the theorem follows from

$$\begin{array}{ccccccccc}
{}_XE^1_{p,q} & = & h_{p+q}(X^p, X^{p-1}) & \xrightarrow{c_*} & \overline{h}_{p+q}(\bigvee S^p) & \xrightarrow{(\Sigma^p)} & \oplus \overline{h}_q(S^0) & = & \mathcal{C}^{cell}_p(X) \otimes h_q(\mathrm{pt}) \\
\downarrow{\scriptstyle f^1_{p,q}} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow & & \downarrow{\scriptstyle \mathcal{C}^{cell}(f)\otimes \mathrm{id}} \\
{}_YE^1_{p,q} & \to & h_{p+q}(Y^p, Y^{p-1}) & \xrightarrow{c_*} & \overline{h}_{p+q}(\bigvee S^p) & \xrightarrow{(\Sigma^p)} & \oplus \overline{h}_q(S^0) & = & \mathcal{C}^{cell}_p(Y) \otimes h_q(\mathrm{pt}).
\end{array}$$

Here, $c\colon X^p \to X^p/X^{p-1}$ denotes the collapsing map and $\bar{f}$ the (well-defined) map induced by f on the quotient. Furthermore, Σ denotes the suspension isomorphism on homology theory, where we identified $h_{p+q}(\bigvee S^p, \mathrm{pt})$ with $\oplus h_{p+q}(S^p, \mathrm{pt})$ with the help of the isomorphisms induced by the inclusions. The suspension isomorphism is natural with respect to continuous maps because it is nothing but the Mayer-Vietoris boundary operator of the excisive triad (SX, C_+X, C_-X), which commutes with $S(f)$, because $S(f)$ is a triad-preserving continuous map. In the right square, $\mathcal{C}^{cell}(f)$ is the matrix $(a_{i,j})$, where $a_{i,j}$ is the degree of the map $S^p_i \hookrightarrow \bigvee S^p \xrightarrow{\bar{f}} \bigvee S^p \twoheadrightarrow S^p_j$. □

The attentive reader certainly has noticed that the commutativity of

$$\begin{array}{ccc} h_*(X^\alpha, X^\beta) & \xrightarrow{j_*} & h_*(X^p, X^q) \\ \downarrow f_* & & \downarrow f_* \\ h_*(Y^\alpha, Y^\beta) & \xrightarrow{j_*} & h_*(Y^p, Y^q), \end{array} \qquad \begin{array}{ccc} h_*(X^\alpha, X^\beta) & \xrightarrow{\Delta} & h_*(X^p, X^q) \\ \downarrow f_* & & \downarrow f_* \\ h_*(Y^\alpha, Y^\beta) & \xrightarrow{\Delta} & h_*(Y^p, Y^q) \end{array}$$

is actually everything we need. Any statement breaks down to the commutativity of these two diagrams. Consequently, a collection of homomorphisms $T_{\alpha\beta}\colon h_*(X^\alpha, X^\beta) \to h_*(Y^\alpha, Y^\beta)$ commuting with all inclusions and all boundary operators yields the same result.

Theorem C.0.19 (AHSS preserves trafos between homology theories)**.** *Let $T\colon h \to h'$ be a natural transformation between two homology theories. For every CW-complex it induces a map between the resulting Atiyah Hirzebruch spectral sequences $T^r_{p,q}$. On the 1-page the map is given by*

$$T^1_{p,q}\colon E^1_{p,q} \cong \mathcal{C}^{cell}_p(X) \otimes h_q(\mathrm{pt}) \xrightarrow{\mathrm{id}\otimes T} \mathcal{C}^{cell}_p(X) \otimes h'_q(\mathrm{pt}).$$

These induced morphisms commutes with the morphisms induced by continuous maps.

Proof. As noted before, this proof is completely analogous to the proof of Theorem C.0.18. Since T is a natural transformation between two homology theories, it will commute with any homomorphism induced by continuous maps. So the second part of the statement also follows. □

Example C.0.20 (relations between different bordisms). There are natural transformation between different bordism theories, namely, forgetting additional structures. Example 1.2.8 shows that a manifold carries a Spin structure if and only if it carries an orientation and a $\mathrm{Pin}^\pm$ structure. So forgetting additional structure gives natural transformations

$$\begin{array}{ccccc} & & \Omega^{\mathrm{Pin}^-}_*(X,A) & & \\ & \overset{\text{FORGET}}{\nearrow} & & \overset{\text{FORGET}}{\searrow} & \\ \Omega^{\mathrm{Spin}}_*(X,A) & & & & \Omega^{\mathrm{O}}_*(X,A) \\ & \underset{\text{FORGET}}{\searrow} & & \underset{\text{FORGET}}{\nearrow} & \\ & & \Omega^{\mathrm{SO}}_*(X,A). & & \end{array}$$

Theorem C.0.21. *If $f\colon X \to Y$ is a cellular map with a cellular section σ, then, for each $k \in \mathbb{Z}$, there is an isomorphism between $h_k(X)$ and $\ker h_k(f) \oplus h_k(Y)$ given by*

$$h_k(X) \xrightarrow[\cong]{\mathrm{id}-h_k(\sigma\circ f)\oplus h_k(f)} \ker h_k(f) \oplus h_k(Y).$$

This isomorphism defines a split on the corresponding Atiyah-Hirzebruch spectral sequences

$$\left(E^r_{p,q}, d^r\right) \xrightarrow{\cong} \left({}^{(1)}E^r_{p,q} \oplus {}^{(2)}E^r_{p,q}, {}^{(1)}d^r \oplus {}^{(2)}d^r\right),$$

where $\left({}^{(2)}E^r_{p,q}, {}^{(2)}d^r\right)$ *is the Atiyah-Hirzebruch spectral sequence induced from* $h_*(Y)$ *and* $\left({}^{(1)}E^r_{p,q}, d^r\right)$ *is a spectral sequence converging to* $\ker h_*(f)$.

Proof. Since the isomorphisms $\mathrm{id} - h_k(\sigma \circ f) = h_k(\mathrm{id}) - h_k(\sigma \circ f)$ are differences of morphisms induced by continuous maps, they commute with all maps induced by inclusions and all boundary operators. Thus, they induce natural splits

$$h_k(X^\alpha, X^\beta) \cong \ker h_k(f) \oplus h_k(Y^\alpha, Y^\beta)$$

and thus the groups in the Atiyah-Hirzebruch spectral sequence split

$$E^r_{p,q} \xrightarrow{\mathrm{id}-(\sigma\circ f)^r_{p,q} \oplus f^r_{p,q}} \ker f^r_{p,q} \oplus {}_Y E^r_{p,q} =\!=\!=\!=\!= {}^{(1)}E^r_{p,q} \oplus {}^{(2)}E^r_{p,q} \; .$$

By definition, ${}^{(2)}d^r$ is the differential of the Atiyah-Hirzebruch spectral sequence of $h_*(Y)$. We define ${}^{(1)}d^r_{p,q}$ to be the restriction of $d^r_{p,q}$ on $\ker f^r_{p,q}$. Since the differentials are natural with respect to continuous maps, ${}^{(1)}d^r_{p,q}$ will have $\ker f^r_{p-r,q+r-1}$ as target space. The same is true for ${}^{(2)}d^r_{p,q}$ for the same reasons. Consequently, we have a split of spectral sequences $\{E^r_{p,q}, d^r\} = \{{}^{(1)}E^r_{p,q} \oplus {}^{(2)}E^r_{p,q}, {}^{(1)}d^r \oplus {}^{(2)}d^r\}$. □

Corollary C.0.22. *Let* $(E^r_{*,*}, d^r)$ *be the Atiyah-Hirzebruch spectral sequence of a connected CW-complex. Then, all differentials with space* $E^r_{0,q}$ *for* $q \in \mathbb{Z}$ *vanish.*

Proof. Since the 0-skeleton $X^{(0)}$ of X is a single point, the unique map $\{\mathrm{pt}\} \to X^{(0)}$ gives an isomorphism between $E^r_{0,q}$ and $h_q(\mathrm{pt})$ which is inverse to the morphism induced by the constant map $\mathrm{const}\colon X \to \{\mathrm{pt}\}$. Therefore $\ker \mathrm{const}^r_{0,q} = \{0\} \subseteq E^r_{0,q}$ and the splitting in Theorem C.0.21 gives

$$(E^r_{*,*}, d^r) \cong ({}_{\mathrm{pt}}E^r_{*,*} \oplus \tilde{E}^r_{*,*}, 0 \oplus \tilde{d}^r),$$

where ${}_{\mathrm{pt}}E^r_{*,*}$ is the Atiyah-Hirzebruch spectral sequence for a single point and $\tilde{E}^r_{p,q} = E^r_{p,q}$, if $p > 1$ and zero otherwise. The result follows. □

If the homology theory carries a product structure, the corresponding Atiyah-Hirzebruch spectral sequence can be equipped with a product structure which approximates the given one [Swi02, p.352ff]. This can be very useful in computations. Note that the spectrum $\mathbf{MPin}^+$ is not a ring spectrum but a module spectrum over **MSpin**. However, for the calculation in chapter 3 we only need that Pin^- bordism is a $\Omega^{\mathrm{Spin}}_*(\mathrm{pt})$-module. So, we sketch the special case where the coefficient ring $k_*(\mathrm{pt})$ of a homology theory with a ring structure acts on the homology theory h_*.

Theorem C.0.23. *If* $k_*(\mathrm{pt}) \curvearrowright h_*(X, A)$ *acts naturally with respect to continuous maps and the boundary operators, then there is a* $k_*(\mathrm{pt})$*-action on the corresponding Atiyah-Hirzebruch spectral sequence approximating the original action. The differentials and the morphisms induced by continuous maps are equivariant with respect to this action this action.*

Under the identification $E^1_{p,q} = \mathcal{C}^{cell}_p(X) \otimes h_*(\mathrm{pt})$*, the action corresponds to the action of* $k_*(\mathrm{pt})$ *on the second factor.*

Proof. Again, the diagrams

$$\begin{array}{ccc} k_*(\mathrm{pt}) \otimes h_*(X^\alpha, X^\beta) & \xrightarrow{\mathrm{id}\otimes j_*} & k_*(\mathrm{pt}) \otimes h_*(X^p, X^q) \\ \downarrow & & \downarrow f_* \\ h_*(X^\alpha, X^\beta) & \xrightarrow{j_*} & h_*(X^p, X^q) \end{array}$$

and

$$\begin{array}{ccc} k_*(\mathrm{pt}) \otimes h_*(X^\alpha, X^\beta) & \xrightarrow{\Delta} & k_*(\mathrm{pt}) \otimes h_*(X^p, X^q) \\ \downarrow & & \downarrow f_* \\ h_*(X^\alpha, X^\beta) & \xrightarrow{\Delta} & h_*(X^p, X^q) \end{array}$$

commute, so the assertion can be checked similarly to the proof of Theorem C.0.18. □

Example C.0.24. If M is a Spin manifold and N a Pin^- manifold, then $M \times N$ is a Pin^- manifold with a canonical Pin^- structure. So, we define a right action

$$\begin{aligned} \Omega_q^{\mathrm{Pin}^-}(X, A) \times \Omega_p^{\mathrm{Spin}}(\mathrm{pt}) &\to \Omega_{p+q}^{\mathrm{Pin}^-}(X, A) \\ [M, \partial M; f] \times [N] &\mapsto [M \times N; f \circ \mathrm{pr}_1]. \end{aligned}$$

The assignment is well-defined because if B is a Spin boundary of $N_1 \sqcup \overline{N_2}$, then $(M \times B; f \circ \mathrm{pr}_1)$ bounds the disjoint union of $(M \times N_1; f \circ \mathrm{pr}_1)$ and $(M \times \overline{N_2}; f \circ \mathrm{pr}_1)$. Recall that a Spin structure and its inverse yields the same Pin^- structure after forgetting the underlying orientation. So, $(M \times \overline{N_2}; f \circ \mathrm{pr}_1)$ and $(M \times N_2; f \circ \mathrm{pr}_1)$ represent the same element in $\Omega_{p+q}^{\mathrm{Pin}^-}(X, A)$ and thus, the action does not depend on the representative Spin manifold. A similar calculation shows that the action does not depend on the choice of the representative singular Pin^- manifold. It it obvious that this action is natural with respect to morphisms induced by continuous maps. The calculation

$$\begin{aligned} \partial[M \times N, \partial M \times N; f \circ \mathrm{pr}_1] &= [\partial(M \times N); \partial(\partial M \times N); f \circ \mathrm{pr}_1|_{\partial(M \times N)}] \\ &= [\partial M \times N; f_{\partial M} \circ \mathrm{pr}_1] \\ &= [N] \times [\partial M; f|_{\partial M}] \\ &= [N] \times \partial[M; f] \end{aligned}$$

shows that the action commutes with the boundary operator. Consequently, $\Omega_*^{\mathrm{Spin}}(\mathrm{pt})$ acts on the Atiyah-Hirzebruch spectral sequence for Pin^- bordism from the right.

This discussion generalises immediately to other bordism theories. For example, $\Omega_*^{\mathrm{O}}(X, A)$ carries a $\Omega_*^{\mathrm{O}}(\mathrm{pt})$ action. The same is true for oriented bordism and Spin bordism.

Prospects

Although the original problem was only to determine the first five Pin^- bordism groups of $B\mathrm{O}(2)$, a natural question arises how the higher groups look like.

It turns out that the tools we have used so far are not appropriate. Indeed, the Atiyah-Hirzebruch spectral sequence, which is very close to the homology theory it approximates, allows a fast and direct computation of the associated graded modules

$$\bigoplus_{p+q=k} E^{\infty}_{p,q}$$

without any complicated auxiliary calculations – at least in small degrees. Using the AHSS, one can show

$$\begin{aligned}\Omega_5^{\mathrm{Pin}^-}(B\mathrm{O}(2)) &\cong \Omega_5^{\mathrm{Pin}^-}(\mathrm{pt}) \oplus \Omega_5^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt}) \oplus \ker \pi_5 \\ &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus G,\end{aligned}$$

where G is a group of order 4. However, the involved calculations are much longer than the calculations in the main text. This suggests that the complexity of the involved calculations should increase dramatically with the degree. The extension problem cannot be solved conceptually either. In this thesis, we used two approaches. The first one was to compare Pin^- bordism with unoriented bordism, a tactic we can use also for higher degrees. However, this tactic does not see cyclic extensions because unoriented bordism is a $\mathbb{Z}_2$-vector space. To verify that $\Omega_2^{\mathrm{Pin}^-}(\mathbb{R}P^\infty, \mathrm{pt})$ is cyclic, we used the classification theorem for surfaces, which only works in dimension two. To determine the isomorphism class of G, we have to be creative.

So, in order to hunt the higher bordism groups, it is advisable to use the Adams spectral sequence. It has the disadvantage that we have to do rather complicated auxiliary calculations to determine the second page and that it is rather ungeometric (it uses only the homotopy type of $\mathbf{MPin}^+$ and not its geometric meaning, therefore finding generators requires additional effort). But its great advantage is that it provides often – after given explicitly all modules of the second page – effective methods for the determination of the associated graded module and for the solutions of the extension problem.

Another interesting question is whether there is a geometric isomorphism

$$\Omega_n^{\mathrm{Pin}^-}(\mathrm{pt}) \to \Omega_{n+1}^{\mathrm{Spin}}(\mathbb{R}P^\infty, \mathrm{pt}).$$

Recall that the isomorphism constructed in Corollary 1.4.12 maps the other way around. A good candidate seems to be the homomorphism in Theorem

1.4.14, but the author could not prove it. Such a diffeomorphism would be very useful for calculations because $\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^\infty)$ has a multiplicative structure.

Danksagung

Wie jedes größere Projekt ist auch diese Masterarbeit nicht frei von Weggefährten, die mich die ganze Zeit begleitet und unterstützt haben.

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Eigenständigkeitserklärung

Hiermit erkläre ich gemäß der Prüfungs- und Studienordnung des konsekutiven Master-Studiengang „Mathematik“ an der Georg-Agust-Universität Göttingen §10 (6) Satz 4, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Göttingen, den 23 August 2017

Thorsten Hertl

Errata

Section 1.5 and 3.2: There are mistakes in the calculation of the Pin^- bordism coefficients. Two statements are wrong, namely Lemma 1.5.8, this is essential, and Lemma 1.5.10, which only affects the determination of the coefficient group $\Omega_4^{\mathrm{Pin}^-}(\mathrm{pt})$ - a group not needed in the calculation in chapter 3. However, the falseness of the proof of Lemma 1.5.8 leaves Lemma 1.5.9 unproven and maybe Lemma 1.5.10, but its proof in wrong nonetheless. The incompleteness affects Theorem 3.2.2, namely $d^2_{5,0} \neq 0$ becomes unproven and, therefore, the determination of $\Omega_4^{\mathrm{Pin}^-}(B\mathrm{O}(2))$ is incomplete too! Luckily, everything except Lemma 1.5.10 can be repaired so far, leaving $\Omega_4^{\mathrm{Pin}}(\mathrm{pt})$ determined, but I have referred to the complete list, so it should be not a big deal.

Lemma C.0.25. *$d^2_{4,1}\colon E^2_{4,1} = H_4(\mathbb{R}P^\infty;\Omega_1^{\mathrm{Spin}}) \to H_2(\mathbb{R}P^\infty;\Omega_2^{\mathrm{Spin}})$ is not zero!*

Proof. Recall that the group $E^2_{p,2} = H_p(\mathbb{R}P^\infty;\Omega_2^{\mathrm{Pin}^-})$ is generated by $\left[\mathbb{R}P^2 \times (S^1_{Lie})^2, \mathrm{pr}_1\right]$. Thus, forgetting the orientation gives an isomorphism ${}_{\mathrm{Spin}}E^2_{i,j} \to {}_{\mathrm{Pin}}E^2_{i,j}$ is an isomorphism for $(i,j) = (4,1)$ or $(2,2)$, see the remark about the forget-functor at the beginning of section 3.2. Therefore, the $d^2_{4,1}$ in the AHSS approximating $\Omega_*^{\mathrm{Spin}}(\mathbb{R}P^\infty)$ is not zero. □

Since Lemma 1.5.8 is repaired, everything else holds as well.

Section 2.4: Corollary 2.4.2 is wrong; the pairs of numbers has to $(2,3)$ and $(0,1)$, respectively. But this is an easy calculation that I have done wrong in the first place.