HOMOTOPICAL ROBUSTNESS OF SYMMETRIES ON THE CAYLEY PLANE

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Abstract. We compute the rational homotopy class of the action $\mathrm{F}_4 \times \mathbb{O}P^2 \to \mathbb{O}P^2$ between the minimal models of source and target. As an application, we derive the effect of the associated inclusion $\mathrm{F}_4 \to \mathrm{hAut}(\mathbb{O}P^2)$ on rational homotopy groups. Although special emphasis is put on the Cayley plane, our results generalise to all simply connected rank 1-symmetric spaces.

1. Introduction

Let $\mathbb{O}P^2$ be the Cayley plane, which is defined to be the subspace of $\mathrm{Herm}(3, \mathbb{O})$ consisting of all hermitian (3×3)-matrices with entries in the octonions $\mathbb{O}$ that are rank one projections. Embed $\mathrm{Herm}(1, \mathbb{O}) = \mathbb{O}$ and $\mathrm{Herm}(2, \mathbb{O})$ into the upper left corner $\mathrm{Herm}(3, \mathbb{O})$, that is via the map $A \mapsto \mathrm{diag}(A, 0)$, and set $\mathbb{O}P^m = \mathbb{O}P^2 \cap \mathrm{Herm}(m + 1, \mathbb{O})$ for $m = 0, 1$.

The compact Lie group F_4 acts faithfully and transitively on $\mathbb{O}P^2$ with stabiliser $\mathrm{Stab}(\mathbb{O}P^0) = \mathrm{Spin}(9)$ and $\mathrm{Stab}(\mathbb{O}P^1) = \mathrm{Spin}(7)$, see [6], especially Theorem 14.79 paired with the definition of the Spin(9) action in the proof of Theorem 14.99. The left action therefore gives rise to an inclusion

$$s\colon \mathrm{F}_4 \to \mathrm{hAut}(\mathbb{O}P^2) \quad \text{and} \quad s\colon \mathrm{F}_4 / \mathrm{Spin}(7) \to \mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2),$$

where $\mathrm{C}(X, Y)$ is the topological space of all continuous maps from X to Y equipped with the compact open topology, and $\mathrm{hAut}(X) \subseteq \mathrm{C}(X, X)$ is the subspace consisting of all homotopy equivalences.

Our first result describes the effect of s on rational homotopy groups.

Theorem A. *The homomorphisms*

$$\pi_r(s)\colon \pi_r(\mathrm{F}_4)_{\mathbb{Q}} \to \pi_r(\mathrm{hAut}(\mathbb{O}P^2), \mathrm{id})_{\mathbb{Q}},$$
$$\pi_r(s)\colon \pi_r(\mathrm{F}_4 / \mathrm{Spin}(7))_{\mathbb{Q}} \to \pi_r(\mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2), \mathrm{incl})_{\mathbb{Q}},$$

are isomorphisms if $r > 11$. The target groups are zero if $r \leq 11$.

Corollary B. *The rational homotopy groups of $\mathrm{hAut}(\mathbb{O}P^2)$ and $\mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2)$ are given by*

$$\pi_r(\mathrm{hAut}(\mathbb{O}P)^2, \mathrm{id})_{\mathbb{Q}} = \pi_r(\mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2), \mathrm{incl})_{\mathbb{Q}} = \begin{cases} \mathbb{Q}, & \text{if } r = 15, 23, \\ 0, & \text{otherwise.} \end{cases}$$

Although Corollary B can be easily computed from Sullivan's technique [10] and its generalisation to arbitrary mapping space in [3], these computations do not provide the information that the non-trivial elements arise from $\pi_{15}(\mathrm{F}_4)_{\mathbb{Q}}$ and $\pi_{23}(\mathrm{F}_4)_{\mathbb{Q}}$.

Similar results have been by established Meier and Strebel for $\mathbb{R}P^n$ [8], by Sasao for $\mathbb{C}P^n$ [9], and by Yamaguchi for $\mathbb{H}P^n$ [11]. However, their proof strategies are not applicable to the Cayley plane. Indeed, Meier and Strebel rely on the classification of $\mathbb{Q}$-acyclic spaces with $\mathbb{Q}$-acyclic fundamental groups, while Sasao and Yamaguchi use the fibrations

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n \quad \text{and} \quad S^3 \to S^{4n+3} \to \mathbb{H}P^n$$

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in an essential manner. The Cayley plane, however, does not exhibit an analogous fibration because the octonions fail to be associative, which forces us to use a proof strategy that is entirely different from theirs.

In contrast to the rather topological approach of Sasao and Yamaguchi, we make use of the full-fledged machinery of rational homotopy theory by transferring the problem into the set-up of Sullivan algebras, where it becomes essentially an (linear) algebraic problem that can be solved fairly hands-on. This approach has the additional advantage that all projective spaces can be treated in a unifying theme, so we automatically provide new proofs for the results in [9] and [11].

Furthermore, our approach allows to derive the stronger result that describes the rational homotopy class of the symmetry group action on projective spaces. To formulate this result in the special case of the symmetry action $\ell\colon \mathrm{F}_4 \times \mathbb{O}P^2 \to \mathbb{O}P^2$ (for the more general case we refer to Proposition 3.6 and Proposition 3.5), let $\mathsf{M}_{\mathbb{O}P^2} = \Lambda[a_8, b_{23} \,|\, db_{23} = a_8^3]$ be the minimal model of $\mathbb{O}P^2$ in the sense of rational homotopy theory, and recall that the rational cohomology ring $H(\mathrm{F}_4) = \Lambda[\xi_3, \xi_{11}, \xi_{15}, \xi_{23}]$ is an exterior algebra.

Theorem C. *The rational model $\mathsf{M}(\ell)\colon \mathsf{M}_{\mathbb{O}P^2} \to H(\mathrm{F}_4) \otimes \mathsf{M}_{\mathbb{O}P^2}$ of the symmetry action $\mathrm{F}_4 \curvearrowright \mathbb{O}P^2$ is the (homotopy class of the) homomorphism of commutative, differential, graded algebras determined by*

$$\mathsf{M}(\ell)(a_8) = 1 \otimes a_8 \quad \textit{and} \quad \mathsf{M}(\ell)(b_{23}) = 1 \otimes b_{23} + \xi_{15} \otimes a_8 + \xi_{23} \otimes 1.$$

Moreover, the coefficients ξ_{15} and ξ_{23} are homotopy invariant in the sense that they remain unchanged if one changes $\mathsf{M}(\ell)$ within its homotopy class.

As a consequence, the rational homotopy class of $s\colon \mathrm{F}_4 \to \mathrm{hAut}(\mathbb{O}P^2)$ is completely determined by its induced homomorphisms on rational homotopy groups (which is not clear at first).

Outline of the article: Section 2 provides a concise recollection on rational homotopy theory needed in this article. Subsection 2.1 presents the abstract theory while Subsection 2.2 provides the rational models for the examples we study in Section 3. It serves further as an opportunity to introduce our notational conventions. Section 3 is reserved for the proofs of Theorem 3.7 and 3.8 that imply Theorem A and Proposition 3.5 and 3.6 that imply Theorem C.

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2. Preliminaries

2.1. **Rational Homotopy Theory.**

We start with a concise recollection of the required basics on rational homotopy theory followed by some sample calculations in the next subsection. More details can be found in [2] and [4]. We first discuss rational spaces and then look at their algebraic counterparts.

2.1.1. *Rational Spaces.* Recall that a path-connected topological space X is called *nilpotent* if its fundamental group is nilpotent and if the action of $\pi_1(X)$ on $\pi_n(X)$ is nilpotent, see Chapter II of [7]. A nilpotent space is called *rational* if, for all n and $r \in \mathbb{N}$, the n-th power map $(\,\cdot\,)^n\colon \pi_r(X) \to \pi_r(X)$ is a bijection (an isomorphism if $r > 1$). The *rationalisation* of a topological space X is a pair $(X_\mathbb{Q}, \mathsf{l}_\mathbb{Q})$ consisting of a rational space $X_\mathbb{Q}$ and a continuous map $\mathsf{l}_\mathbb{Q}\colon X \to X_\mathbb{Q}$ such that precomposition with $\mathsf{l}_\mathbb{Q}$ induces a bijection

$$(\,\cdot\,) \circ \mathsf{l}_\mathbb{Q}\colon [X_\mathbb{Q}, Y] \xrightarrow{\cong} [X, Y]$$

for all nilpotent, rational spaces Y. Rationalisations exists for every nilpotent space, see Theorem 3A in Chapter II of [7], and their universal property implies that they are unique up to homotopy equivalence. Moreover, $\pi_r(\mathsf{l}_{\mathbb{Q}})\colon \pi_r(X)_{\mathbb{Q}} \to \pi_r(X_{\mathbb{Q}})$ induces an isomorphism[1], see Theorem 3B in Chapter II of [7].

A continuous map $f\colon X \to Y$ between two path-connected nilpotent topological spaces is called a *rational homotopy equivalence* if it induces an isomorphism between their rationalised homotopy groups. Two spaces are called *rational homotopy equivalent* if there exists a zig-zag $X \to Z_1 \leftarrow \cdots \leftarrow Z_n \to Y$ of rational homotopy equivalences between them. Equivalently, X and Y are rational homotopy equivalent if their rationalisations are weak homotopy equivalent.

The reason why we consider larger class of nilpotent spaces (instead of simply connected spaces) is that this subclass of topological spaces is closed under taking mapping spaces: If X is a finite CW complex and Y a nilpotent CW complex, then each path-component of the space of continuous maps $\mathrm{C}(X,Y)$ is nilpotent, see Theorem 2.5 in Chapter II of [7]. For our purpose, the following result is essential.

Theorem 2.1. *[7, Theorem 3.11] Let X be a finite, connected CW-complex, Y be a nilpotent CW-complex, and $\mathsf{l}_{\mathbb{Q}}\colon Y \to Y_{\mathbb{Q}}$ its rationalisation. Then postcomposition with $\mathsf{l}_{\mathbb{Q}}$ induces a rationalisation $\mathsf{l}_{\mathbb{Q}} \circ (\,\cdot\,)\colon \mathrm{C}(X,Y;f) \to \mathrm{C}(X,Y_{\mathbb{Q}},\mathsf{l}_{\mathbb{Q}} \circ f)$, where $\mathrm{C}(X,Y;f)$ is the path component of $\mathrm{C}(X,Y)$ that contains $f\colon X \to Y$.*

The advantage of rational spaces is that their homotopy types and the set of their homotopy classes can be completely described in terms of algebraic data.

2.1.2. *Sullivan Algebras.* The algebraic counterpart for rational topological spaces will be the category of commutative, differential, (cohomologically) graded, (unital) algebras (cdgas in short) over the field $\mathbb{Q}$ that are concentrated in non-negative degrees. The assignment sinding a topological space X to $\Omega_{PL}(S(X))$, the cdga of polynomial differential forms on the singular set $S(X)$, yields a contravariant functor $\mathsf{Top} \to \mathsf{CDGA}$. The cohomology $H(\Omega_{PL}(X))$ is isomorphic to $H_{sing}(X;\mathbb{Q})$, see [10] or [4, Theorem 10.9].

The role of a CW-complex inside the category CDGA are taken by *Sullivan algebras*. Recall that a Sullivan algebra $(\Lambda V, d)$ is the free algebra of a (non-negatively) graded (rational) vector space V such that d satisfies the *nilpotency* condition, which means that there is a filtration $V(0) \subseteq V(1) \subseteq \dots$ such that $\bigcup V(k) = V$ and such that $d = 0$ on $V(0)$ and $d\colon V(n) \to \Lambda V(n-1)$ for all $n \geq 1$. A Sullivan algebra is called *minimal* if $d(v)$ is quadratic, that is $d(V) \subseteq \Lambda^{\geq 2} V$. A homomorphism of commutative, differential, graded algebras (abbreviated to dga-homomorphism in the future) $\varphi\colon (A,d) \to (B,d)$ is a *quasi-isomorphism* if its induced homomorphism on cohomology $H(\varphi)\colon H(A,d) \to H(B,d)$ is an isomorphism.

An important example of a (non-minimal) Sullivan algebra is the *algebraic interval* $I = \Lambda[t, dt]$, where t is a variable in degree zero and the differential satisfies the tautological relation $d(t) = dt$ and $d(dt) = 0$. It comes with two dga-homomorphisms $\mathrm{ev}_0, \mathrm{ev}_1\colon \Lambda[t,dt] \to \mathbb{Q}$ that are completely described by $\mathrm{ev}_j(t) = j$ for $j = 0, 1$. A *homotopy* between two dga-homomorphisms $\varphi_0, \varphi_1\colon A \to B$ is a dga-homomorphism $H\colon A \to I \otimes B$ such that $\mathrm{ev}_j \circ H = \varphi_j$. Being homotopic is an equivalence relation if the domain is a Sullivan algebra, see [2, Proposition 6.3]. Moreover, there is a form of Whitehead's theorem in the sense that postcomposition with quasi-isomorphisms $\varphi\colon (A_1,d) \to (A_2,d)$ induces a bijection between homomotopy classes $[\Lambda V, A_1] \to [\Lambda V, A_2]$, see [2, Proposition 5.7].

A *Sullivan model* of a topological space X is a pair of a Sullivan algebra $(\Lambda V, d)$ and a quasi-isomorphism $m\colon \Lambda V \to \Omega_{PL}(X)$. If $(\Lambda V, d)$ is minimal, then we refer to it as a *minimal model* of X and denote it with M_X. The notation is justified, because minimal models exist [4, Proposition 14.3] and are unique up to isomorphism [4, Theorem 14.12]. A *rational model* for a continuous map $f\colon X \to Y$ is a dga-homomorphism between Sullivan models of domain and target $\varphi\colon \Lambda V_Y \to \Lambda V_X$ such that $m_X \circ \varphi$ is homotopic to $\Omega(f) \circ m_Y$. By Whiteheads theorem, the homotopy of this model

[1]For $\pi_1(X)$, we use the Maclev-completion, which agrees with the tensor product if $\pi_1(X)$ is abelian, see Chapter I of [7] for details.

is independent of the choice of Sullivan models used to define it. If we use minimal models, we will use the notation $\mathsf{M}(f)$ (for the homotopy class or any of its representatives).

It was proved in [2, Theorem 4.3] that CDGA has model category structure with weak-equivalences being quasi-isomorphisms and fibrations being surjective dga-homomorphism and that Sullivan algebras are cofibrant objects. The authors of [2] further show, see Chapter 9 in loc. cit. that the assignment $X \mapsto \mathsf{M}_X$ gives rise to a Quillen adjunction

$$\langle \cdot \rangle \colon \mathsf{CDGA} \rightleftarrows \mathsf{Top} : \mathsf{M}_{(\,\cdot\,)}$$

that induces an equivalence of categories

$$\mathrm{fin}_{\mathbb{Q}}\text{-}\mathrm{Ho}(\mathsf{CDGA}) \overset{\cong}{\rightleftarrows} \mathrm{fin}_{\mathbb{Q},\mathrm{nil}}\text{-}\mathrm{Ho}(\mathsf{Top})$$

where $\mathrm{fin}_{\mathbb{Q}}$-Ho($\mathsf{CDGA}$) is the full subcategory of Ho(CDGA) whose objects are quasi-isomorphic to a minimal algebra ΛV with a generating graded vector space that is finite dimensional in each degree. $\mathrm{fin}_{\mathbb{Q},\mathrm{nil}}$-Ho($\mathsf{Top}$) is the full subcategory of Ho(Top) whose objects are weakly equivalent to a rational nilpotent space of finite $\mathbb{Q}$-type, that is, its rational homology is finite dimensional in each degree. In particular, we have the following consequence, see [2, Proof 11.9] for second part of the following statement.

Theorem 2.2. *For every two rational, nilpotent spaces of finite $\mathbb{Q}$-type X, Y, we have a bijection of homotopy classes*

$$[X, Y] \cong [\mathsf{M}_Y, \mathsf{M}_X] \quad \textit{given by} \quad [f] \mapsto [\mathsf{M}_f]. \tag{2.1}$$

Furthermore, if the minimal model $\mathsf{M}_Y = \Lambda V_Y$ is generated by V_Y, then there is a natural bijection $\pi_k(Y)_{\mathbb{Q}} \cong \mathrm{Hom}(V_Y^k, \mathbb{Q})$, which is an isomorphism whenever the domain is abelian.

For a dga-homomorphism $\varphi\colon A \to B$ between two cdga and $r \in \mathbb{Z}$, denote by $\mathrm{Der}_r^{\varphi}(A, B)$ the vector space of all φ-derivations that lower the degree by r, that is, the vector space of all linear maps $\theta\colon A \to B$ that lower the degree by r and satisfy $\theta(a_1 \cdot a_2) = \theta(a_1)\varphi(a_2) + (-1)^{r\cdot\deg(a_2)}\varphi(a_1)\cdot\theta(a_2)$. It is straighforward to verify that for $\theta \in \mathrm{Der}_r^{\varphi}(A, B)$ the linear map $\delta_r(\theta) := d\theta - (-1)^r\theta d$ is again a φ-derivation and that $\delta_{r-1} \circ \delta_r = 0$. In particular, δ turns $\mathrm{Der}^{\varphi}(A, B) = \bigoplus_{r\in\mathbb{Z}} \mathrm{Der}_r^{\varphi}(A, B)$ into a homologically graded chain complex. The next result generalises the one of Sullivan [10] from homotopy automorphisms to arbitrary mapping spaces and was proved in [3].

Theorem 2.3 (Theorem 1 in [3])**.** *Let X, Y be nilpotent CW complexes with X finite and Y of finite $\mathbb{Q}$-type. Then there are isomorphisms of $\mathbb{Q}$-vector spaces*

$$\pi_r(\mathrm{C}(X, Y), f)_{\mathbb{Q}} \cong \mathrm{Der}_r^{\mathsf{M}(f)}(\mathsf{M}_Y, \mathsf{M}_X, \delta),$$
$$\Gamma\pi_1(\mathrm{C}(X, Y), f)_{\mathbb{Q}} \cong \mathrm{Der}_1^{\mathsf{M}(f)}(\mathsf{M}_Y, \mathsf{M}_X, \delta),$$

where $\Gamma\pi_1(\mathrm{C}(X, Y), f)_{\mathbb{Q}}$ denotes the rational vector space $\bigoplus_j \Gamma_j/\Gamma_{j+1} \otimes \mathbb{Q}$ associated to the lower central series $\Gamma_0 = \pi_1(\mathrm{C}(X, Y), f) \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_n = \{1\}$ of the nilpotent group $\pi_1(\mathrm{C}(X, Y), f)$.

2.2. **Examples of Rational Models.** We work out the rational models for the spaces we consider Section 3. Unless stated otherwise, rational coefficients are understood.

Example 2.4. The minimal model of S^n with $n = 2k + 1$ is given by $\mathsf{M}_{S^n} = \Lambda[a_n \,|\, da_n = 0]$. It is easy to see that a map $m\colon \mathsf{M}_{S^n} \to \Omega(S^n)$ that sends a_n to a generator of the volume form $\mathrm{vol}_{S^n} = H^n(S^n) = H^n(\Omega(S^n), d)$ is a quasi-isomorphism.

Example 2.5. The minimal model of S^n with $n = 2k$ is given by $\mathsf{M}_{S^n} = \Lambda[a_n, b_{2n-1} \,|\, db_{2n-1} = a_n^2]$. A model map is inductively constructed as follows: Send a_n to a generator ω_n of the volume form $\mathrm{vol}_{S^n} \in H^n(S^n)$. We know that ω_n^2 must be exact, so we send b_{2n-1} to a form η with $d\eta = \omega_n^2$. By construction, the map m extends to a quasi-isomorphism $m\colon \mathsf{M}_{S^n} \to \Omega(S^n)$.

Example 2.6. The cohomology ring of $\mathbb{C}P^n$ is well known: $H(\mathbb{C}P^n;\mathbb{Q}) = \mathbb{Q}[a_2]/\langle a_2^{n+1}\rangle$. Therefore, the minimal model of $\mathbb{C}P^n$ is given by $\mathsf{M}_{\mathbb{C}P^n} = \Lambda[a_2, b_{2n+1} \,|\, db_{2n+1} = a_2^{n+1}]$. A model map $m\colon \mathsf{M}_{\mathbb{C}P^n} \to \Omega(\mathbb{C}P^n)$ sends a_2 to a generator α_2 of $a_2 \in H^2(\mathbb{C}P^n)$ and b_{2n+1} to a primitive of α_2^{n+1}.

Since the $\mathbb{C}P^n$ can be given a CW-structure such that the $(2m)$-skeleton $(\mathbb{C}P^n)^{(2m)}$ can be identified with $\mathbb{C}P^m$ it follows that the inclusion $\mathbb{C}P^m \hookrightarrow \mathbb{C}P^n$ induces the canonical projection $\mathbb{Q}[a_2]/\langle a_2^{n+1}\rangle \to \mathbb{Q}[a_2]/\langle a_2^{m+1}\rangle$ on cohomology. It is not hard to see that there is one and only one dga-homomorphism $\mathsf{M}_{\mathbb{C}P^n} \to \mathsf{M}_{\mathbb{C}P^m}$ that induces the aforementioned ring homomorphism on cohomology, namely

$$\mathsf{M}_{\mathbb{C}P^n} = \Lambda[a_2, b_{2n+1} \,|\, db_{2n+1} = a_2^{n+1}] \to \Lambda[a_2, b_{2m+1} \,|\, db_{2n+1} = a_2^{m+1}] = \mathsf{M}_{\mathbb{C}P^m},$$
$$a_2 \mapsto a_2, \quad b_{2n+1} \mapsto a_2^{n-m} b_{2m+1}.$$

Example 2.7. Completely analogously to the previous example, we derive

$$\mathsf{M}_{\mathbb{H}P^n} = \Lambda[a_4, b_{4n+3} \,|\, db_{4n+3} = a_4^{n+1}] \to \Lambda[a_4, b_{4n+3} \,|\, db_{4m+3} = a_4^{m+1}] = \mathsf{M}_{\mathbb{H}P^m},$$
$$a_4 \mapsto a_4, \quad b_{4n+3} \mapsto a_4^{n-m} b_{4m+3}.$$

Example 2.8. It is well known that $\mathbb{O}P^1 \cong S^8$ and that $\mathbb{O}P^2$ has a CW-structure of the form $\{*\} = \mathbb{O}P^0 \subset \mathbb{O}P^1 = (\mathbb{O}P^2)^{(8)} \subseteq \mathbb{O}P^2 = (\mathbb{O}P^2)^{(16)}$. In particular, its cohomology ring is given by $H(\mathbb{O}P^n) = \mathbb{Q}[a_8]/\langle a_8^{n+1}\rangle$ for $n \le 2$. Arguing as in Example 2.6, we deduce that the inclusion $\mathbb{O}P^1 \hookrightarrow \mathbb{O}P^2$ is modelled by

$$\mathsf{M}_{\mathbb{O}P^2} = \Lambda[a_8, b_{23} \,|\, db_{23} = a_2^3] \to \Lambda[a_8, b_{15} \,|\, db_{15} = a_2^2] = \mathsf{M}_{\mathbb{O}P^1}$$
$$a_8 \mapsto a_8, \quad b_{23} \mapsto a_8 b_{15}.$$

By a Theorem of Hopf the rational cohomology of a compact Lie group is a free exterior algebra, see [5, Theorem 1.34], so its cohomology group agrees with its minimal model. To set up our notational conventions, we recall the cohomology rings of the Lie groups of our interests, see Example 3.37, and Example 3.40-3.42 in [5] for the equivalent statements of their classifying spaces.

Example 2.9.

(i) If $n = 2k$ is even, then $\mathsf{M}_{\mathrm{SO}(n)} = H(\mathrm{SO}(n)) = \Lambda[\pi_3, \ldots, \pi_{4k-5}, \varepsilon_{2k-1}]$. Note in particular that $\mathsf{M}_{\mathrm{SO}(2)} = \Lambda[\varepsilon_1]$.
(ii) If $n = 2k+1$ is odd, then $\mathsf{M}_{\mathrm{SO}(n)} = H(\mathrm{SO}(n)) = \Lambda[\pi_3, \ldots, \pi_{4k-1}]$
(iii) For all $n \ge 1$ we have $\mathsf{M}_{\mathrm{U}(n)} = H(\mathrm{U}(n)) = \Lambda[\gamma_1, \ldots, \gamma_{2n-1}]$.
(iv) For all $n \ge 1$ we have $\mathsf{M}_{\mathrm{Sp}(n)} = H(\mathrm{Sp}(n)) = \Lambda[q_3, \ldots, q_{4n-3}]$.
(v) Since $\mathrm{Spin}(9) \to \mathrm{SO}(9)$ is a two-sheeted cover, the two groups are rational homotopy equivalent and we have $\mathsf{M}_{\mathrm{Spin}(9)} = \Lambda[\pi_3, \pi_7, \pi_{11}, \pi_{15}]$. Similarly we deduce that $\mathsf{M}_{\mathrm{Spin}(7)} = \Lambda[\pi_3, \pi_7, \pi_{11}]$.
(vi) Borel [1] computed the integral cohomology ring of F_4. By tensoring with the rational numbers, we deduce that $\mathsf{M}_{\mathrm{F}_4} = H(\mathrm{F}_4) = \Lambda[\xi_3, \xi_{11}, \xi_{15}, \xi_{23}]$.

It is well known that inclusion of subgroups induce the obvious homomorphisms between their homology groups, for example the inclusion $\mathrm{SO}(2k-1) \to \mathrm{SO}(2k)$ induces the algebra homomorphism $\Lambda[\pi_3, \ldots, \pi_{4k-5}, \varepsilon_k] \to \Lambda[\pi_3, \ldots, \pi_{4k-5}]$ that sends π_j to π_j and ε_{2k-1} to zero. In particular, the homomorphism induced by the inclusion $\mathrm{Spin}(9) \to \mathrm{F}_4$ sends ξ_{23} to zero and $\xi_j \mapsto \pi_j$ for $j = 3, 11, 15$.

Next we would like to understand the rational model of the quotient maps. We start with a technical lemma that classifies homotopies between certain Sullivan algebras.

Lemma 2.10. *For $k \ge 1$ and $n \ge 2$, consider a minimal Sullivan algebra of the form $(\Lambda V, d) = \Lambda[a_{2k}, b_{2nk-1} \,|\, db_{2kn-1} = a_{2k}^n]$ and let $(A, 0)$ be a cdga with trivial differential and generated by elements of odd degree. Then each homotopy class has a unique representative.*

Proof. Each homotopy H is uniquely determined its image of the generators. Degree reasons imply

$$H(a_{2k}) = dt \otimes x_1 \qquad \text{and} \qquad H(b_{2kn-1}) = x_{2kn-1} + t \otimes y_{2kn-1} + dt \otimes z_{2kn-2}.$$

Since $da_{2k} = 0$ and the differential on the target vanishes, the algebra homomorphism is a dga-homomorphism if and only if

$$0 = (dt \otimes x_1)^n = H(db_{2kn-1}) = dH(b_{2kn-1}) = dt \otimes y_{2kn-1},$$

which is equivalent to $y_{2nk-1} = 0$. It follows that $\mathrm{ev}_0 \circ H = \mathrm{ev}_1 \circ H$, as claimed. □

Lemma 2.11. *The evaluation map* $\mathrm{ev}_{e_1}\colon \mathrm{SO}(n+1) \to S^n$ *has the following rational models:*

(i) If $n = 2k$*, then* $\mathsf{M}(\mathrm{ev}_{e_1})\colon \mathsf{M}_{S^n} \to \mathsf{M}_{\mathrm{SO}(n+1)}$ *is induced by* $a_n \mapsto 0$ *and* $b_{2n-1} \mapsto \pi_{4k-1}$*.*

(ii) If $n = 2k+1$*, then* $\mathsf{M}(\mathrm{ev}_{e_1})\colon \mathsf{M}_{S^n} \to \mathsf{M}_{\mathrm{SO}(n+1)}$ *is given by* $a_{2k+1} \mapsto \varepsilon_{2k+1}$*.*

Proof. The odd case is a well-known statement about rational cohomology of Lie groups, so we will prove the even-dimensional case only. The dga-homomorphism $\mathsf{M}(\mathrm{ev}_{e_1})$ modelling the evaluation map must be induced by

$$a_{2k} \mapsto \mathrm{dec}_{2k} \qquad \text{and} \qquad b_{4k-1} \mapsto \lambda_{4k-1}\pi_{4k-1} + \mathrm{dec}_{4k-1}$$

with $\lambda_{4k-1} \in \mathbb{Q}$ and decomposable elements $\mathrm{dec}_{2k} \in H^{2k}(\mathrm{SO}(2k+1)) = H^{2k}(\mathrm{SO}(2k))$ and $\mathrm{dec}_{4k-1} \in H^{4k-1}(\mathrm{SO}(2k))$. Since the composition $\mathrm{SO}(2k) \hookrightarrow \mathrm{SO}(2k+1) \xrightarrow{\mathrm{ev}_1} S^{2k}$ is the constant map, this forces dec_{2k} and dec_{4k-1} to vanish. The induced long exact sequence of rational homotopy groups together with their algebraic description in terms of their minimal models, Theorem 2.2 and Example 2.9 implies that λ_{4k-1} must be non-zero. By rescaling π_{4k-1} if necessary, we may assume $\lambda_{4k-1} = 1$. □

The next example concerns complex projective spaces.

Lemma 2.12. *The map* $\mathrm{ev}_{[e_0]}\colon \mathrm{U}(n+1) \to \mathbb{C}P^n$ *given by evaluating at* $[e_0] = [1 : 0 : \cdots : 0]$ *is modelled by the dga-homomorphism*

$$\mathsf{M}(\mathrm{ev}_{[e_0]})\colon \mathsf{M}_{\mathbb{C}P^n} \to \mathsf{M}_{\mathrm{U}(n+1)} \qquad \textit{induced by} \qquad a_2 \mapsto 0 \textit{ and } b_{2n+1} \mapsto \gamma_{2n+1}.$$

Proof. By applying the long exact sequence of rational homotopy groups to the fibration $\mathrm{U}(1) \times \mathrm{U}(n) \to \mathrm{U}(n+1) \xrightarrow{\mathrm{ev}_{[e_0]}} \mathbb{C}P^n$, we deduce $\pi_{2n+1}(\mathrm{ev}_{[e_0]})_{\mathbb{Q}}$ must be an isomorphism. From the natural isomorphism $\pi_n(X)_{\mathbb{Q}} \cong \mathrm{Hom}(V_X^n, \mathbb{Q})$ for each nilpotent space of finite $\mathbb{Q}$-type, where V_X is the generating graded vector space of the minimal model of X, we deduce that $\mathrm{ev}_{[e_0]}$ is modelled by a homotopy class of a dga-homomorphism of form[2]

$$a_2 \mapsto 0 \qquad \text{and} \qquad b_{2n+1} \mapsto \gamma_{2n+1} + \mathrm{dec}_{2n+1},$$

where $\mathrm{dec}_{2n+1} \in \Lambda[\gamma_1, \ldots, \gamma_{2n-1}]$ is a linear combination of decomposable elements.

We prove by induction that $\mathrm{dec} = 0$. Starting with $n = 1$, we see $\mathrm{dec}_{2n+1} = 0$ for algebraic reasons.

Evaluation at $[e_0]$ is stable in the sense that the left square in following diagram commutes strictly, so right one of its rational models commutes up to homotopy

$$\begin{array}{ccc} \mathrm{U}(n+1) & \xrightarrow{\mathrm{ev}_{[e_0]}} & \mathbb{C}P^n \\ {\scriptstyle \mathrm{incl}}\uparrow & & \uparrow \\ \mathrm{U}(n) & \xrightarrow{\mathrm{ev}_{[e_0]}} & \mathbb{C}P^{n-1} \end{array} \qquad \begin{array}{ccc} \mathsf{M}_{\mathrm{U}(n+1)} & \xleftarrow{\mathsf{M}(\mathrm{ev}_{[e_0]})} & \mathsf{M}_{\mathbb{C}P^n} \\ {\scriptstyle \mathsf{M}(\mathrm{incl})}\downarrow & & \downarrow {\scriptstyle \mathsf{M}(\mathrm{ev}_{[e_0]})\; b_{2n+1} \mapsto a_2 b_{2n-1}} \\ \mathsf{M}_{\mathrm{U}(n)} & \xleftarrow{\mathsf{M}(\mathrm{ev}_{[e_0]})} & \mathsf{M}_{\mathbb{C}P^{n-1}} \end{array}$$

where $\mathsf{M}(\mathrm{incl})$ is given by $\gamma_j \mapsto \gamma_j$ and $\gamma_{2n+1} \mapsto 0$. By Lemma 2.10 the right diagram must commute, so we deduce $0 = \mathsf{M}(\mathrm{incl}) \circ \mathsf{M}(\mathrm{ev}_{[e_0]}) = \mathsf{M}(\mathrm{incl})(\gamma_{2n+1} + \mathrm{dec}_{2n+1}) = \mathrm{dec}_{2n+1}$. □

[2] After rescaling γ_{2n+1} if necessary.

The argument of the previous proof generalises to the quaternionic projective spaces and the Cayley plane.

Lemma 2.13. *The evaluation maps* $\mathrm{ev}_{[e_0]}\colon \mathrm{Sp}(n+1) \to \mathbb{H}P^n$ *and* $\mathrm{ev}_{[e_0]}\colon \mathrm{F}_4 \to \mathbb{O}P^2 = \mathrm{F}_4 / \mathrm{Spin}(9)$ *are modelled by the dga-homomorphisms*

$$\begin{aligned} \mathsf{M}_{\mathbb{H}P^n} &\to \mathsf{M}_{\mathrm{Sp}(n+1)} && \textit{induced by} && a_4 \mapsto 0 \ \textit{ and } \ b_{4n+3} \mapsto q_{4n+1}, \\ \mathsf{M}_{\mathbb{O}P^2} &\to \mathsf{M}_{\mathrm{F}_4} && \textit{induced by} && a_8 \mapsto 0 \ \textit{ and } \ b_{23} \mapsto \xi_{23}. \end{aligned}$$

Finally, we would like to calculate the rational homotopy groups of the mapping spaces $\mathrm{C}(\mathbb{K}P^m, \mathbb{K}P^n)$ and $\mathrm{C}(S^m, S^n)$ using Theorem 2.3. We begin with discussing the case of projective spaces.

Lemma 2.14. *For* $1 \le m \le n$, $k = \dim_{\mathbb{C}}(\mathbb{K})$*, and the standard inclusion* $\mathrm{incl}\colon \mathbb{K}P^m \hookrightarrow \mathbb{K}P^n$*, the rational homotopy groups of the mapping spaces* $\mathrm{C}(\mathbb{K}P^m, \mathbb{K}P^n)$ *are given by*

$$\pi_r(\mathrm{C}(\mathbb{K}P^m, \mathbb{K}P^n), \mathrm{incl})_{\mathbb{Q}} \cong \begin{cases} \mathbb{Q}, & \textit{if } r \in \{2k(n-\alpha)+2k+1 \,:\, 0 \le \alpha \le m\}, \\ 0, & \textit{else.} \end{cases}$$

Proof. Recall that the minimal models of $\mathbb{K}P^n$ is given by $\Lambda[a_{2k}, b_{2k(n+1)-1} \,|\, db_{2k(n+1)-1} = a_{2k}^{n+1}]$ and that the inclusion is modelled by the dga-homomorphism $\mathsf{M}(\mathrm{incl})$ that sends a_{2k} to a_{2k} and $b_{2k(n+1)-1}$ to $a_{2k}^{n-m} b_{2k(m+1)-1}$.

Since derivations with domain a free algebra are uniquely determined by the images of the basis of the generating vector space, we deduce that the vector space $\mathrm{Der}(\mathsf{M}_{\mathbb{K}P^n}, \mathsf{M}_{\mathbb{K}P^m})$ of all positive graded $\mathsf{M}(\mathrm{incl})$-derivation is generated by the derivations

$$1 \otimes a_{2k}^{\vee}, \qquad a_{2k}^{\alpha} \otimes b_{2k(n+1)-1}^{\vee}, \qquad a_{2k}^{\beta} b_{2k(m+1)-1} \otimes b_{2k(n+1)-1}^{\vee}$$

with $0 \le \alpha \le n$ and $0 \le \beta < n-m$. The derivations are of degree 2, $2k(n-\alpha)+2k+1$, and $2k(n-m)-2k\beta$, respectively.

Straightforward computations yield the identities

$$\delta(1 \otimes a_{2k}^{\vee}) = -(n+1) a_{2k}^{n} \otimes b_{2k(n+1)-1}^{\vee}, \qquad \delta(a_{2k}^{\alpha} \otimes b_{2k(n+1)-1}^{\vee}) = 0,$$
$$\delta(a_{2k}^{\beta} b_{2k(m+1)-1} \otimes b_{2k(n+1)-1}^{\vee}) = a_{2k}^{\beta+m+1} \otimes b_{2k(n+1)-1}^{\vee}.$$

We conclude that $\ker \delta$ is spanned by the derivations $a_{2k}^{\alpha} \otimes b_{2k(n+1)-1}$ and that the image of δ is generated by the derivations $a_{2k}^{\alpha} \otimes b_{2k(n+1)-1}$ with $m < \alpha \le n$. The resulting homology groups are then generated by $a_{2k}^{\alpha} \otimes b_{2k(n+1)-1}^{\vee}$ for $0 \le \alpha \le m < n$, so we deduce

$$H_r(\mathrm{Der}(\mathsf{M}_{\mathbb{K}P^n}, \mathsf{M}_{\mathbb{K}P^m}), \delta) = \begin{cases} \mathbb{Q}, & \text{if } r \in \{2k(n-\alpha)+2k+1 \,:\, 0 \le \alpha \le m\} \\ 0, & \text{else.} \end{cases}$$

The claim now follows from 2.3. □

The corresponding result for spheres can be obtained in the same fashion. However, if $m < n$ we have the following alternative, because, in this case, the inclusion $S^m \hookrightarrow S^n$ is nullhomotopic, so the fibration $\mathrm{C}_*(S^m, S^n) \to \mathrm{C}(S^m, S^n) \xrightarrow{\mathrm{ev}_{e-1}} S^n$ with the fibre the space $\mathrm{C}_*(S^m, S^n)$ of all base-point preserving continuous maps $S^m \to S^n$ has a section that sends a point x to the constant map $\mathrm{const}_x\colon S^m \to S^n$ with value x. From this, we deduce

$$\pi_k(\mathrm{C}(S^m, S^n))_{\mathbb{Q}} = \pi_k(S^n)_{\mathbb{Q}} \oplus \pi_k(\Omega^m S^n)_{\mathbb{Q}} = \pi_k(S^n)_{\mathbb{Q}} \oplus \pi_{k+m}(S^n)_{\mathbb{Q}},$$

which allows us to read off the rational homotopy groups without calculations. We conclude the next result.

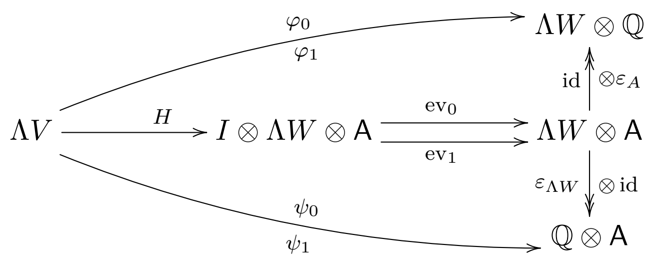
Lemma 2.15. *For* $1 \le m \le n$*, the rational homotopy groups* $\pi_k(\mathrm{C}(S^m, S^n), \mathrm{incl})_{\mathbb{Q}}$ *are given as follows:*

(i) *If* n *is odd, then* $\pi_k(\mathrm{C}(S^m, S^n), \mathrm{incl})_{\mathbb{Q}} = \mathbb{Q}$ *if* $k = n$ *or* $k = n - m > 0$ *and zero otherwise.*

(ii) *If n is even and $m < n$, then $\pi_k(\mathrm{C}(S^m, S^n), \mathrm{incl})_{\mathbb{Q}} = \mathbb{Q}$ if $k \in \{n, 2n-1\}$ or if $k \in \{n-m, 2n-1-m\}$ is positive.*
(iii) *If $n = m$ even, then $\pi_k(\mathrm{hAut}(S^n), \mathrm{id})_{\mathbb{Q}} = \mathbb{Q}$ if $k = 2n-1$.*

3. Robustness of Symmetries

We now prove the main results of this article by calculating rational models for the actions $\ell\colon \mathrm{Sym}(M, g) \times M \to M$. The basis for the proofs will be the following technical lemma, which should be considered as a generalisation of Lemma 2.10. To formulate it, recall that a cdga A is *connected* if $\mathsf{A}^0 = \mathbb{Q} \cdot 1$ and that it is *augmented* if there exists a dga-homomorphism $\varepsilon_{\mathsf{A}}\colon \mathsf{A} \to \mathbb{Q}$. Recall further that an arbitrary homotopy $H\colon \Lambda V \to \Lambda[t, dt] \otimes \Lambda W \otimes \mathsf{A}$ with A an augmented cdga gives rise to the following commutative diagram, in which we abbreviate $\Lambda[t, dt]$ to I:



Proposition 3.1. *For $k \geq 1$ and $n \geq m \geq 2$, let $(\Lambda V, d) = \Lambda[a_{2k}, b_{b_{2kn-1}} \,|\, db_{2kn-1} = a_{2k}^n]$ and $(\Lambda W, d) = \Lambda[a_{2k}, b_{2mk-1} \,|\, db_{2mk-1} = a_{2k}^m]$ be two Sullivan algebras and let $(\mathsf{A}, 0)$ be an augmented, connected cdga with trivial differential. Then each homotopy $H\colon (\Lambda V, d) \to \Lambda[t, dt] \otimes (\Lambda W, d) \otimes \mathsf{A}$ is of the form*

$$H(a_{2k}) = 1 \otimes \lambda_0 \cdot a_{2k} \otimes 1 + 1 \otimes 1 \otimes \psi_0(a_{2k}) + dt \otimes x_{2k-1},$$

$$\begin{aligned} H(b_{2nk-1}) = 1 \otimes \Bigg(& \lambda_0^{n-m} a_{2k}^{n-m} b_{2km-1} + 1 \otimes \psi_0(b_{2kn-1}) + h_0(b_{2kn-1}) \\ & + \sum_{\alpha=0}^{n-m-1} (\lambda_0 a_{2k})^{\alpha} b_{2km-1} \otimes \psi_0(a_{2k})^{n-m-\alpha} \Bigg) \\ & + t \otimes \left(n \cdot \psi_0(a_{2k})^{n-1} x_{2k-1} + n \cdot \sum_{\alpha=1}^{n-1} (\lambda_0 a_{2k})^{\alpha} \otimes \psi_0(a_{2k})^{n-\alpha-1} x_{2k-1} + dh_{dt}(b_{2kn-1}) \right) \\ & + dt \otimes h_{dt}(b_{2kn-1}), \end{aligned}$$

where[3] *$x_{2k-1} \in \mathsf{A}^{2k-1}$ is closed, $\psi(a_{2k})^r = 0$ for $r > n-m$, and $h_0(b_{2kn-1}) \in \Lambda^{\geq 1} W \otimes \mathsf{A}^{\geq 1}$ is closed while $h_{dt}(b_{2kn-1}) \in \Lambda W \otimes \mathsf{A}$ satisfies $dh_{dt}(b_{2kn-1}) \in a_{2k}^m \cdot \Lambda W \otimes \mathsf{A}$* [4].

We mostly apply this proposition in the case where $\psi_0 = 0$, so we explicitly spell out the simplified formula.

Corollary 3.2. *If, in addition, $\psi_0 = 0$ in Proposition 3.1, then all possible homotopies are of the form*

$$\begin{aligned} H(a_{2k}) &= 1 \otimes \lambda_0 \cdot a_{2k} \otimes 1 + dt \otimes x_{2k-1} \\ H(b_{2kn-1}) &= 1 \otimes (\lambda_0^{n-m} a_{2k}^{n-m} b_{2km-1} + h_0(b_{2kn-1})) + \\ & \quad t \otimes (n \lambda_0^{n-1} a_{2k}^{n-1} x_{2k-1} + dh_{dt}(b_{2kn-1})) + dt \otimes h_{dt}(b_{2kn-1}) \end{aligned}$$

[3] We implicitly assume that H is a dga-homomorphism, of course, so that $d\psi(b_{2kn-1}) = \psi_0(a_{2k})^n = 0$.
[4] Note that if $m = n$, then this condition implies that $dh_{dt}(b_{2kn-1}) = 0$.

with $x_{2k-1} \in \Lambda(W \otimes A)^{2k-1}$ *closed,* $h_{dt}(b_{2kn-1}) \in (\Lambda W \otimes A)^{2kn-2}$ *an element with* $dh_{dt}(b_{2kn-1}) \in a_{2k}^m \cdot (\Lambda W \otimes \mathsf{A})^{2k(n-m)-1}$ *and* $h_0(b_{2kn-1}) \in \Lambda^{\geq 1} W \otimes \mathsf{A}^{\geq 1}$.

Remark 3.3. Informally speaking, the degree of freedom we have to change a dga-homomorphism $\Phi_0 = \mathrm{ev}_0 \circ H$ within its homotopy class is the coefficients behind the t-factor. Proposition 3.1 tells us that homotopies are quite rigid, because the only can only perturb Φ_0 by an the element $x_{2k-1} \in \mathsf{A}$ and an exact element $dh_{b_{2kn-1}} \in a_{2k}^m \cdot (\Lambda W \otimes \mathsf{A})^{2k(n-m)-1}$ (which is automatically zero if $n = m$).

Proof. We can uniquely decompose $H(a_{2k})$ and $H(b_{2kn-1})$ as follows:

$$\begin{aligned} H(a_{2k}) &= 1 \otimes (\varphi_0(a_{2k}) \otimes 1 + 1 \otimes \psi_0(a_{2k})) + \\ &\quad t \otimes ((\varphi_1 - \varphi_0)(a_{2k}) \otimes 1 + 1 \otimes (\psi_1 - \psi_0)(a_{2k})) + dt \otimes x_{2k-1} \\ H(b_{2kn-1}) &= 1 \otimes (\varphi_0(b_{2kn-1}) \otimes 1 + 1 \otimes \psi_0(b_{2kn-1}) + h_0(b_{2kn-1})) + \\ &\quad t \otimes ((\varphi_1 - \varphi_0)(b_{2kn-1}) + 1 \otimes (\psi_1 - \psi_0)(b_{2kn-1}) + h_t(b_{2kn-1})) + \\ &\quad dt \otimes h_{dt}(b_{2kn-1}), \end{aligned}$$

for some elements $x_{2kn-1} \in \mathsf{A}^{2kn-1}$, $h_0(b_{2kn-1}), h_t(b_{2kn-1}) \in \Lambda^{\geq 1} W \otimes \mathsf{A}^{\geq 1}$, and $h_{dt}(b_{2kn-1}) \in (\Lambda W \otimes \mathsf{A})^{2kn-2} = \mathbb{Q} \otimes \mathsf{A}^{2kn-2}$. Conversely, every such assignment determines a unique algebra homomorphism $H\colon \Lambda V \to I \otimes \Lambda W \otimes \mathsf{A}$ and H is a dga-homomorphism if and only if $H(da_{2k}) = dH(a_{2k})$ as well as $H(db_{2kn-1}) = dH(b_{2kn-1})$ are satisfied.

The first equation translates into

$$0 = H(da_{2k}) = dH(a_{2k}) = dt \otimes ((\varphi_1 - \varphi_0)(a_{2k}) \otimes 1 + 1 \otimes (\psi_1 - \psi_0)(a_{2k}) + dx_{2k}),$$

which is equivalent to

$$\varphi_0(a_{2k}) = \varphi_1(a_{2k}) = \lambda_0 a_{2k}, \qquad \psi_0(a_{2k}) = \psi_1(a_{2k}), \qquad \text{and} \qquad dx_{2k-1} = 0. \tag{3.1}$$

for some $\lambda_0 \in \mathbb{Q}$.

The left hand side of the second identity is now given by

$$\begin{aligned} H(db_{2kn-1}) = H(a_{2k})^n &= \sum_{\alpha=0}^{n} \binom{n}{\alpha} \cdot 1 \otimes \varphi_0(a_{2k})^\alpha \otimes \psi_0(a_{2k})^{n-\alpha} \\ &\quad + n \sum_{\alpha=0}^{n-1} \binom{n-1}{\alpha} dt \otimes \varphi_0(a_{2k})^\alpha \otimes \psi_0(a_{2k})^{n-\alpha-1} x_{2k-1}, \end{aligned} \tag{3.2}$$

while the right hand side is given by

$$\begin{aligned} dH(b_{2kn-1}) = \;& 1 \otimes (d\varphi_0(b_{2kn-1}) \otimes 1 + 1 \otimes d\psi_0(b_{2kn-1}) + dh_0(b_{2kn-1})) \\ & + t \otimes (d(\varphi_1 - \varphi_0)(b_{2kn-1}) \otimes 1 + d(\psi_1 - \psi_0)(b_{2kn-1}) \otimes 1 + dh_t(b_{2kn-1})) \\ & + dt \otimes ((\varphi_1 - \varphi_0)(b_{2kn-1}) \otimes 1 + 1 \otimes (\psi_1 - \psi_0)(b_{2kn-1}) + h_t(b_{2kn-1}) - dh_{dt}(b_{2kn-1})) \end{aligned}$$

By degree reasons, $h_{dt}(b_{2kn-1}) \in (\Lambda W \otimes \mathsf{A})^{2nk-2} = (\Lambda[a_{2k}, b_{2km-1}] \otimes \mathsf{A})^{2kn-2}$, so $dh_{dt}(b_{2kn-1}) \in a_{2k}^m \cdot (\Lambda W \otimes \mathsf{A})$. In particular, if $n = m$, then $(\Lambda W \otimes \mathsf{A})^{2nk-2} = (\Lambda[a_{2k}] \otimes \mathsf{A})^{2kn-2}$ and $h_{dt}(b_{2kn-1})$ must be closed.

$$\varphi_0(a_{2k})^n = d\varphi_0(b_{2kn-1}), \quad d\psi_0(b_{2kn-1}) = \psi_0(a_{2k})^n \quad \text{and} \quad dh_0(b_{2kn-1}) = 0, \tag{3.3}$$

$$dh_0(b_{2kn-1}) = \sum_{\alpha=1}^{n-1} \binom{n}{\alpha} \varphi_0(a_{2k})^\alpha \otimes \psi_0(a_{2k})^{n-\alpha} \tag{3.4}$$

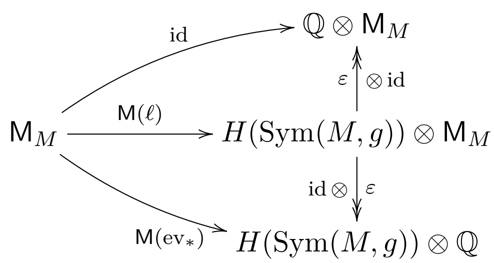
$$\varphi_1(b_{2kn-1}) = \varphi_0(b_{2kn-1}) \qquad \text{and} \qquad \psi_1(b_{2kn-1}) = \psi_0(b_{2kn-1}) + n \cdot \psi_0(a_{2k})^{n-1} x_{2k-1}, \tag{3.5}$$

$$h_t(b_{2kn-1}) = n \sum_{\alpha=1}^{n-1} \binom{n-1}{\alpha} \varphi_0(a_{2k})^\alpha \otimes \psi_0(a_{2k})^{n-\alpha-1} \cdot x_{2k-1} + dh_{dt}(b_{2kn-1}) \tag{3.6}$$

Note that the first condition of (3.3) implies that $\varphi_0(b_{2kn-1}) = \varphi_0(a_{2k})^{n-m} b_{2km-1}$. Furthermore, observe that (3.4) is a condition; it forces the right-hand side to be exact, and this holds true if and only if either $\varphi(a_{2k}) = 0$ or if $\psi_{a_{2k}}^{n-\alpha} = 0$ for all $\alpha < m$ because $a_{2k}^m = db_{2km-1} \in \Lambda W$.

Plugging (3.2) - (3.6) and the just derived observations into the defining equation of $H(a_{2k})$ and $H(b_{2kn-1})$, we obtain the claimed formulas. □

Corollary 3.4. *If (M, g) is a simply connected symmetric space of rank 1 and $\mathrm{Sym}(M, g)$ its symmetry group that is considered in Example 2.9, then the rational model $\mathsf{M}(\ell)$ of its symmetric action ℓ makes the following diagram commutative*



Proof. Since ℓ restricts to $\mathrm{ev}_* = \ell(\,\cdot\,, *)$ on $\mathrm{Sym}(M, g) \times \{*\}$ and to the identity on $\{1\} \times M$, we know that the diagram in the statement must commutative up to homotopy. Since the augmentation maps are surjective, they satisfy the the homotopy lifting property [2, Proposition 5.3]. After changing $\mathsf{M}(\ell)$ in its homotopy class if necessary, we assume that the lower triangle commutes strictly.

Since all rank 1-symmetric spaces have a minimal model of the form $\Lambda[a_{2k}, b_{2kn-1} \,|\, db_{2kn-1} = a_{2k}^n]$, we can apply Proposition 3.1 with $\mathsf{A} = \mathbb{Q}$ and deduce that every dga-homomorphism is unique in its homotopy class (because in the notation of Proposition 3.1 $x_{2k-1} = 0$ for degree reasons). Thus, the upper triangle strictly commutes, too. □

We begin with the description of the rational model for left action $\ell\colon \mathrm{SO}(n+1) \times S^m \to S^n$.

Proposition 3.5. *The rational model $\mathsf{M}(\ell)\colon \mathsf{M}_{S^n} \to \mathsf{M}_{\mathrm{SO}(n+1)} \otimes \mathsf{M}_{S^m}$ is the zero map if $m < n$. If $m = n$, then the rational model is given by the homotopy class of the dga-homomorphism*

$$\begin{aligned} a_{2k} &\mapsto 1 \otimes a_{2k}, \qquad b_{4k-1} \mapsto 1 \otimes b_{4k-1} + \pi_{4k-1} \otimes 1, && \text{if } n = 2k, \\ a_{2k+1} &\mapsto 1 \otimes a_{2k+1} + \varepsilon_{2k+1} \otimes 1, && \text{if } n = 2k+1. \end{aligned}$$

Proof. We deal with the even dimensional case first. Lemma 2.11 and Corollary 3.4 imply that a dga-homomorphism φ_0 modelling ℓ must send a_{2k} to $1 \otimes a_{2k}$ and $b_{4k-1} \mapsto 1 \otimes b_{4k-1} + \pi_{4k-1} \otimes 1 + \mathrm{dec}_{2k-1} \otimes a_{2k-1}$ with $\mathrm{dec}_{4k-1} \in H^{2k-1}(\mathrm{SO}(2n+1)) = H^{2k-1}(\mathrm{SO}(2n))$. By applying Proposition 3.1 with $\mathsf{A} = H(\mathrm{SO}(2n+1))$, $x_{2k-1} = -\mathrm{dec}_{2k-1}/2$ we find a homotopy between φ_0 and $\varphi_1 = \mathsf{M}(\ell)$ in the claim.

In the odd case, each rational model φ_0 for ℓ must be of the form $\varphi(a_{2k+1}) = 1 \otimes a_{2k+1} + \varepsilon_{2k+1} \otimes 1 + \mathrm{dec}_{2k+1} \otimes 1$ with $\mathrm{dec}_{2k+1} \in H^{2k+1}(\mathrm{SO}(2k+1))$. Lemma 2.11 implies now that $\mathrm{dec}_{2k+1} \otimes 1 = 0$. □

Next we have a look at the left action $\ell\colon \mathrm{Sym}(\mathbb{K}P^n, g_{FS}) \times \mathbb{K}P^m \to \mathbb{K}P^n$.

Proposition 3.6. *The left action ℓ is modelled by the (homotopy class) of the dga-homomorphisms $\mathsf{M}(\ell)\colon \mathsf{M}_{\mathbb{K}P^n} \to H(\mathrm{Sym}(\mathbb{K}P^n, g_{FS})) \otimes \mathsf{M}_{\mathbb{K}P^n}$ induced by*

$$\begin{aligned}
&\mathsf{M}(\ell)(a) = 1 \otimes a, && \text{for } \mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\},\\
&\mathsf{M}(\ell)(b_{2n+1}) = 1 \otimes a_2^{n-m} b_{2n+1} + \sum_{\alpha=n-m}^{n} \gamma_{2\alpha+1} \otimes a_2^{n-\alpha}, && \text{for } \mathbb{K} = \mathbb{C},\\
&\mathsf{M}(\ell)(b_{4n+3}) = 1 \otimes a_4^{n-m} b_{4n+3} + \sum_{\alpha=n-m}^{n} q_{4\alpha+3} \otimes a_4^{n-\alpha}, && \text{for } \mathbb{K} = \mathbb{H},\\
&\mathsf{M}(\ell)(b_{8n+7}) = 1 \otimes a_8^{n-m} b_{8n+7} + \sum_{\alpha=n-m}^{n} \xi_{8\alpha+7} \otimes a_8^{n-\alpha}, && \text{for } \mathbb{K} = \mathbb{O}.
\end{aligned}$$

Moreover, the coefficients are homotopy-invariant: they remain unchanged if we replace $\mathsf{M}(\ell)$ by any homotopic representative.

Proof. We spell out the proof for $\mathbb{H}P^n$ only as the proofs for the other cases only differ in notation. We will prove the statement by induction over n for the special case $n = m$ and then deduce the more general case $m \leq n$.

We begin with $n = 1$. For degree reasons and Lemma 2.12, the dga homomorphism $\mathsf{M}(\ell)$ must be of the form $a_4 \mapsto 1 \otimes a_4$ and $b_7 \mapsto 1 \otimes b_7 + q_7 \otimes 1 + \mu \cdot q_3 \otimes a_4$ for some $\mu \in \mathbb{Q}$.

Since $\deg(q_3) = \deg(a_4) - 1$, we may choose $x_3 = -\mu/2 \cdot q_3 \otimes 1$ in Proposition 3.1 to change $\mathsf{M}(\ell)$ in its homotopy classes, to gauge $\mu = 0$, which is not possible for the other coefficients.

For $n > 1$, we argue inductively. As for $\mathbb{H}P^1$, we know that ℓ must have the form

$$a_{4k} \mapsto 1 \otimes a_{4k} \qquad \text{and} \qquad b_{4k+3} \mapsto 1 \otimes b_{4n+3} + \sum_{\alpha=0}^{n} {}_n\zeta_\alpha \otimes a_4^{n-\alpha},$$

with ${}_n\zeta_\alpha \in H^{4\alpha+3}(\mathrm{Sym}(\mathbb{H}P^n); \mathbb{Q})$. By Corollary 3.2, we see that all coefficients except ${}_n\zeta_0$ are homotopy invariant and as in the special case $n = 1$ we can gauge it away if we wish.

To determine the coefficients, we consider the following diagram that commutes up to homotopy

$$\begin{array}{ccc}
\mathsf{M}_{\mathbb{H}P^n} & \longrightarrow & H(\mathrm{Sym}(\mathbb{H}P^n, g_{FS})) \otimes \mathsf{M}_{\mathbb{H}P^n} \\
{\scriptstyle \mathsf{M}(\mathrm{incl})}\downarrow & & \downarrow{\scriptstyle H(\iota)\otimes \mathsf{M}(\mathrm{incl})} \\
\mathsf{M}_{\mathbb{H}P^{n-1}} & \longrightarrow & H(\mathrm{Sym}(\mathbb{H}P^{n-1}, g_{FS})) \otimes \mathsf{M}_{\mathbb{H}P^{n-1}}.
\end{array}$$

Applying Corollary 3.2 with the cdgas $\Lambda W = \mathsf{M}_{\mathbb{H}P^{n-1}}$ and $\mathsf{A} = H(\mathrm{Sym}(\mathbb{H}P^{n-1}, g_{FS}))$ and $\psi_0 = 0$, we see that coefficients ${}_{n-1}\zeta_\alpha$ remain homotopy-invariant except for ${}_{n-1}\zeta_1$. From $\mathsf{M}(\ell) \circ \mathsf{M}(\mathrm{incl}) \simeq H(\iota) \otimes \mathsf{M}(\mathrm{incl})$ and the homotopy invariance of the coefficients ${}_n\zeta_\alpha$ and ${}_{n-1}\zeta_\alpha$ for $(\alpha \neq 0)$ we deduce that

$$(H(\iota) \otimes \mathsf{M}(\mathrm{incl})) \circ \mathsf{M}(\ell)(b_{4n+3}) = 1 \otimes a_4 b_{4n-1} + \sum_{\alpha=0}^{n-1} {}_n\zeta_\alpha \otimes a_4^{n-\alpha}$$

must agree (up to a summand of the form $\mu \cdot \zeta_0 \otimes a_4^{n-1} = \mu \cdot q_3 \otimes a_4^{n-1}$) with

$$(H(\iota) \otimes \mathsf{M}(\mathrm{incl})) \circ \mathsf{M}(\ell)(b_{4n+3}) = 1 \otimes a_4 b_{4n-1} + \sum_{\alpha=0}^{n-1} {}_{n-1}\zeta_\alpha \otimes a_4^{n-\alpha}.$$

By comparing coefficients we derive the recursive equation ${}_n\zeta_\alpha = {}_{n-1}\zeta_\alpha$ which determines all coefficients ${}_n\zeta_\alpha$ with $0 < \alpha < n$ to be ${}_n\zeta_\alpha = q_{4\alpha+3}$. Lemma 2.13 determines the top coefficient ${}_n\zeta_n \in H^{4n+3}(\mathrm{Sym}(\mathbb{H}P^n, g_{FS}))$ to satisfy ${}_n\zeta_n = q_{4n+3}$, and we have proven the statement for $m = n$.

To obtain the general case, we use the homotopy-commutative diagram

$$\begin{array}{ccc} \mathsf{M}_{\mathbb{H}P^n} & \xrightarrow{\mathsf{M}(\ell)} & H(\mathrm{Sym}(\mathbb{H}P^n, g_{FS}))\otimes \mathsf{M}_{\mathbb{H}P^n} \\ & \searrow^{\mathsf{M}(\ell)} & \downarrow{\scriptstyle \mathrm{id}\otimes \mathsf{M}(\mathrm{incl})} \\ & & H(\mathrm{Sym}(\mathbb{H}P^n, g_{FS}))\otimes \mathsf{M}_{\mathbb{H}P^{n-1}} \end{array}$$

that in this case $\mathsf{M}(\ell)\colon \mathsf{M}_{\mathbb{H}P^n} \to H(\mathrm{Sym}(\mathbb{H}P^n))\otimes \mathsf{M}_{\mathbb{H}P^m}$ is still induced by the assignment

$$a_4 \mapsto a_4 \quad \text{and} \quad b_{4n+3} \mapsto \sum_{\alpha=0}^{n} q_{4\alpha+3}\otimes a_4^{n-\alpha}.$$

However, since $a_4^{n-\alpha}$ is exact for $\alpha < m$, we have more gauge freedom, and can choose the $H\colon \mathsf{M}_{\mathbb{H}P^n} \to \Lambda[t,dt]\otimes H(\mathrm{Sym}(\mathbb{H}P^n, g_{FS})\otimes \mathsf{M}_{\mathbb{H}P^m}$ that is given by

$$\begin{aligned} H(a_4) &= 1\otimes 1\otimes a_4 \\ H(b_{4n+3}) &= 1\otimes a_4^{n-m}b_{4m+3} + \sum_{\alpha=0}^{n} q_{4\alpha+3}\otimes a_4^{n-\alpha} \\ &\qquad - t\otimes \sum_{\alpha=0}^{m-1} q_{4\alpha+3}\otimes a_4^{n-\alpha} - dt\otimes \sum_{\alpha=0}^{m-1} q_{4\alpha+3}\otimes a_4^{n-\alpha-(m+1)}b_{4m+3}, \end{aligned}$$

to obtain the claimed formula. □

We finally in the position to prove the main results of this article.

Theorem 3.7. *For all $1 \le m \le n$, the inclusions $s\colon \mathrm{Sym}(\mathbb{K}P^n, g_{FS})/\mathrm{Stab}(\mathbb{K}P^m) \to \mathrm{C}(\mathbb{K}P^m, \mathbb{K}P^n)$ induced by the left action induce isomorphisms on rational homotopy groups in the following degrees:*

$$\begin{aligned} &\pi_k(s)\colon \pi_k(\mathrm{U}(n+1)/(\mathrm{U}(1)\times \mathrm{U}(n-m))_{\mathbb{Q}} \xrightarrow{\cong} \pi_k(\mathrm{C}(\mathbb{C}P^m, \mathbb{C}P^n), \mathrm{incl})_{\mathbb{Q}} && \textit{for all } k>0, \\ &\pi_k(s)\colon \pi_k(\mathrm{Sp}(n+1)/(\mathbb{Z}_2\times \mathrm{Sp}(n-m))_{\mathbb{Q}} \xrightarrow{\cong} \pi_k(\mathrm{C}(\mathbb{H}P^m, \mathbb{H}P^n), \mathrm{incl})_{\mathbb{Q}} && \textit{for all } k>3, \\ &\pi_k(s)\colon \pi_k(\mathrm{F}_4)_{\mathbb{Q}} \xrightarrow{\cong} \pi_k(\mathrm{hAut}(\mathbb{O}P^2), \mathrm{id})_{\mathbb{Q}} && \textit{for all } k>11, \\ &\pi_k(s)\colon \pi_k(\mathrm{F}_4/\mathrm{Spin}(7))_{\mathbb{Q}} \xrightarrow{\cong} \pi_k(\mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2), \mathrm{incl})_{\mathbb{Q}} && \textit{for all } k>0. \end{aligned}$$

In the remaining cases, the target groups are zero and the kernels of the homomorphisms are given by $\pi_3(\mathrm{Sp}(n+1)/(\mathbb{Z}_2\times \mathrm{Sp}(n-m)))_{\mathbb{Q}} \cong \mathbb{Q}$ and $\pi_3(\mathrm{F}_4)_{\mathbb{Q}} \cong \pi_{11}(\mathrm{F}_4)_{\mathbb{Q}} \cong \mathbb{Q}$.

Proof. From Example 2.9 and Lemma 2.14, we see that rational homotopy groups of the left and the right hand side are isomorphisms in the claimed degrees and that the target groups are zero otherwise. The kernel can now easily be read of from Example 2.9. It remains to show that the inclusion a.k.a the adjoint of the symmetry group action induces these isomorphisms.

We spell the proof in detail for the fourth case, that is for the map $\mathrm{F}_4/\mathrm{Spin}(7) \to \mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2)$, but we do in a way that allows a straightforward adaptation to the other cases. For $k = 2, 3$, the homotopy groups $\pi_{8k-1}(\mathrm{F}_4)_{\mathbb{Q}} \cong \mathbb{Q}$, and by Theorem 2.2 we see that a generator f_{8k-1} can be modelled by the dga-homomorphism $\mathsf{M}(g_{8k-1}) = a_{8k-1}\otimes \xi_{8k-1}^{\vee}\colon H(\mathrm{F}_4) \to \mathsf{M}_{S^{8k-1}} = \Lambda[a_{8k-1}]$ that sends ξ_{8k-1} to a_{8k-1} and the other generators to zero.

By Theorem 2.1 composition with the rationalisation-map $\mathsf{l}_{\mathbb{Q}}\colon \mathbb{O}P^2 \to \mathbb{O}P^2_{\mathbb{Q}}$ induces a rationalisation $\mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2; \mathrm{incl}) \to \mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2_{\mathbb{Q}}; \mathsf{l}_{\mathbb{Q}}\circ \mathrm{incl})$. By the exponential law and the universal property of a $\mathbb{Q}$-localisation, each composition

$$S^{8k-1} \xrightarrow{f} \mathrm{F}_4 \overset{s}{\hookrightarrow} \mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2; \mathrm{incl}) \xrightarrow{\mathsf{l}_{\mathbb{Q}}\circ(\,\cdot\,)} \mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2_{\mathbb{Q}}; \mathsf{l}_{\mathbb{Q}}\circ \mathrm{incl})$$

adjoins to the following composition

$$S^{8k-1}_{\mathbb{Q}} \times \mathbb{O}P^1_{\mathbb{Q}} \xrightarrow{f_{\mathbb{Q}} \times \mathrm{incl}_{\mathbb{Q}}} \mathrm{F}_{4,\mathbb{Q}} \times \mathbb{O}P^1_{\mathbb{Q}} \xrightarrow{\ell_{\mathbb{Q}}} \mathbb{O}P^2_{\mathbb{Q}}.$$

Its homotopy class is modelled by the homotopy class of the composition of dga-homomorphism

$$\mathsf{M}_{\mathbb{O}P^2} \xrightarrow{\mathsf{M}(\ell)} H(\mathrm{F}_4) \otimes \mathsf{M}_{\mathbb{O}P^1} \xrightarrow{\mathsf{M}(f)\otimes \mathrm{id}} \mathsf{M}_{S^{8k-1}} \otimes \mathsf{M}_{\mathbb{O}P^1}.$$

From Proposition 3.6 we deduce that under this identification the composition $\mathsf{I}_{\mathbb{Q}} \circ s \circ g_{8k-1}$ is represented by the dga-homomorphism determined by $a_8 \mapsto 1 \otimes a_8$ and $b_{23} \mapsto 1 \otimes a_8 b_{15} + a_{8k-1} \otimes a_8^{3-k}$, while the composition with the constant map $S^{8k-1} \xrightarrow{\mathrm{const}_1} \mathrm{F}_4 \to \mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2_{\mathbb{Q}}; \mathsf{I}_{\mathbb{Q}} \circ \mathrm{incl})$ is represented by the homotopy class of the dga-homomorphism given by $a_8 \mapsto 1 \otimes a_8$ and $b_{23} \mapsto 1 \otimes a_8 b_{15}$, the rational model of the inclusion $\mathrm{incl}\colon \mathbb{O}P^1 \hookrightarrow \mathbb{O}P^2$. For $k = 2, 3$, the elements a_8^{3-k} is not exact in $\mathsf{M}_{\mathbb{O}P^1}$, and we deduce from Proposition 3.1 that the two dga-homomorphisms are *not* homotopic.

We conclude that $s\colon \mathrm{F}_4 \to \mathrm{C}(\mathbb{O}P^1, \mathbb{O}P^2)$ induces a non-zero map between their rational homotopy groups in all degrees where the target is non-zero; by dimension reasons the map is therefore an isomorphism. Since s factors through $\mathrm{F}_4 / \mathrm{Stab}(\mathbb{O}P^1) = \mathrm{F}_4 / \mathrm{Spin}(7)$, which has isomorphic rational homotopy groups to F_4 in these degrees, the induced map remains an isomorphism. □

The same proof-technique applies to the case of spheres. In contrast to projective spaces, here we have homotopy automorphisms that do not arise from the symmetry group of the target.

Theorem 3.8. *For all $1 \leq m \leq n$, the homomorphism $\pi_r(s)\colon \pi_r(\mathrm{SO}(n+1))_{\mathbb{Q}} \to \pi_r(\mathrm{hAut}(S^n))_{\mathbb{Q}}$ induced by canonical inclusion $s\colon \mathrm{SO}(n+1) \to \mathrm{C}(S^m, S^n)$ is the zero map unless $r = n = m$, in which case it is an isomorphism.*

Proof. As in the previous proof, whether or not the map $S^r \to \mathrm{SO}(n+1) \xrightarrow{s} \mathrm{C}(S^m, S^n; \mathrm{incl})$ represents the zero element in $\pi_r(\mathrm{C}(S^m, S^n))_{\mathbb{Q}}$ can be read off from the homotopy class of the dga homomorphism

$$\mathsf{M}_{S^n} \xrightarrow{\mathsf{M}(\ell)} H(\mathrm{SO}(n+1)) \otimes \mathsf{M}_{S^m} \xrightarrow{\mathsf{M}(f)\otimes \mathrm{id}} \mathsf{M}_{S^r} \otimes \mathsf{M}_{S^m},$$

which is the zero map if $m < n$. If $m = n$ is odd, then by Proposition 3.5 the composition is the dga-homomorphism determined by $a_n \mapsto 1 \otimes a_n + \mathsf{M}(f)(\varepsilon_n) \otimes 1$, which is not homotopic to the identity if $[f] \in \pi_n(\mathrm{SO}(n+1))_{\mathbb{Q}}$ represents the element $\varepsilon_n^\vee$.

If $m = n$ is odd, then by Proposition 3.5 the composition is the dga-homomorphism determined by $a_n \mapsto 1 \otimes a_n$ and $b_{2n-1} \mapsto 1 \otimes b_{2n-1} + \mathsf{M}(f)(\pi_{2n-1}) \otimes 1$, which is not homotopic to the identity if $[f] \in \pi_{2n-1}(\mathrm{SO}(n+1))_{\mathbb{Q}}$ has the linear map $\pi_{2n-1}^\vee$ as rational model. □

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