FIBREWISE ORBIFOLD RESOLUTIONS WITH APPLICATIONS TO G$_2$-MODULI SPACES

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Abstract. By resolving the singularities of tailor-made orbifolds via twisted families of blow-ups, we construct manifold bundles $M \to E \to S^2$. Using tools from real homotopy theory, we show that these bundles determine a free subgroup in $\pi_2(B\mathrm{hAut}(M)_0)$. The proof relies on a generalisation of Sullivan's result, which describes the real homotopy groups of the monoid of homotopy automorphisms $\mathrm{hAut}(X)$ in terms of derivations of the minimal model of X, to the monoid $\mathrm{hAut}_A(X)$ of relative homotopy automorphisms.

As an application, we prove that the moduli space of torsion-free G$_2$-structures arising from many generalised Kummer constructions contains a free subgroup of positive rank in its second homotopy group.

1. Introduction

A G$_2$-structure is a differential form $\varphi \in \Omega^3(M)$ on a seven-dimensional manifold M^7 that is *positive* in the sense that the symmetric bilinear form g_φ defined implicitly by the formula

$$g_\varphi(v,w) \cdot \mathrm{vol}_{g_\varphi} = \iota_v(\varphi) \wedge \iota_w(\varphi) \wedge \varphi$$

is positive definite. If φ is parallel with respect to the Levi-Civita connection of its underlying metric g_φ, then we call φ *torsion-free* and the pair (M,φ) a G$_2$-manifold because the holonomy group $\mathrm{Hol}(g_\varphi)$ is then contained in G$_2$. For closed manifold, full holonomy $\mathrm{Hol}(g_\varphi) = \mathrm{G}_2$ is attained if and only if in addition the fundamental group $\pi_1(M)$ is finite [17, Proposition 10.2.2].

The group of diffeomorphisms $\mathrm{Diff}(M)$ and its path-component of the identity $\mathrm{Diff}(M)_0$ act on the space of all torsion-free G$_2$-structures $\mathcal{G}_2^{\mathrm{tf}}(M)$ acts via pull back. In [15], Joyce not only produced the first example of a closed G$_2$-manifold, but also proved that the *moduli space* $\mathcal{G}_2^{\mathrm{tf}}(M)/\mathrm{Diff}(M)_0$ of each closed seven-dimensional manifold is itself a smooth manifold[1] of dimension $b^3(M)$, the third Betti number of M, by showing that the period map $\mathcal{G}_2^{\mathrm{tf}}(M)/\mathrm{Diff}(M)_0 \to H^3(M;\mathbb{R})$, which sends an equivalence class of a torsion-free G$_2$-structure to its de Rham cohomology class, is a local diffeomorphism.

In contrast to the moduli spaces of K3-surfaces, where the analogous period map is an embedding with an explicitly described image, Joyce's result is essentially of local nature and cannot be used to deduce much about the global topological features of G$_2$-moduli spaces. Thus, understanding global topological properties of G$_2$-moduli spaces have become an active field of investigation over the last decade, see [9], [8], [21], and [7]; see also [23] for a recent development on the fundamental group of the full quotient $\mathcal{G}_2^{\mathrm{tf}}(M)/\mathrm{Diff}(M)$.

The main purpose of this article is to study G$_2$-moduli spaces from the view point of real homotopy theory. Our first main result shows that G$_2$-moduli spaces of generalised Kummer constructions in the sense of Joyce [17, Chapter 11] often have non-trivial real second homotopy groups.

Theorem A. *Let M^7 be a simply connected generalised Kummer construction obtained from T^7/Γ by resolving its singularities. Let* N *be the number of path components S of the singular set of T^7/Γ that satisfy the following two properties:*

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[1] If M is not a G$_2$-manifold, then the moduli space is empty, of course.

(i) S has a tubular neighbourhood isometrically isomorphic to $T^3 \times D^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts antipodally on D^4.

(ii) There is a S' different from S satisfying (i) such that their real homology classes $[S], [S'] \in H_3(T^7/\Gamma; \mathbb{R})$ generate the same vector space.

Then

$$\pi_2(\mathcal{G}_2^{\mathrm{tf}}(M)/\mathrm{Diff}(M)_0) \otimes \mathbb{R} \supseteq \mathbb{R}^{\mathrm{N}}.$$

The conditions in Theorem A are easy to check in practice. We indicate this by discussing in Section 5 below some of Joyce's examples in [16]. The results are summarised in the following table:

Examples of Generalised Kummer Constructions from [16]						
Example Number	3	4	6	7	9	11
$\mathbb{Z}^{\mathrm{N}} \subseteq \pi_2(\mathcal{G}_2^{\mathrm{tf}}(M)/\mathrm{Diff}(M)_0)$	$\mathbb{Z}^{12}$	$\mathbb{Z}^{8}$	$\mathbb{Z}^{4}$	$\mathbb{Z}^{2}$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{4}$

The main challenge in constructing non-trivial maps into the moduli space $\mathcal{G}_2^{\mathrm{tf}}(M)/\mathrm{Diff}(M)_0$ stems from the fact it does not have a universal family. To remedy this defect, the author and his collaborators introduced in [7] the *homotopy moduli space* $\mathcal{G}_2^{\mathrm{tf}}(M)/\!\!/\mathrm{Diff}(M)_0$. This space has the universal property that every family of G_2-manifolds parametrised by a base space B (up to a suitable notion of equivalence) gives rise to a unique homotopy class of continuous maps $B \to \mathcal{G}_2^{\mathrm{tf}}(M)/\!\!/\mathrm{Diff}(M)_0$. It further comes with two comparison maps $\mathcal{G}_2^{\mathrm{tf}}(M)/\!\!/\mathrm{Diff}(M)_0 \to B\mathrm{Diff}(M)$ and $\mathcal{G}_2^{\mathrm{tf}}(M)/\!\!/\mathrm{Diff}(M)_0 \to \mathcal{G}_2^{\mathrm{tf}}(M)/\mathrm{Diff}(M)_0$, the latter inducing an injective homomorphism on all higher homotopy groups, see [7, Theorem B].

With the homotopy moduli space at hand, we can sketch a proof of Theorem A: The construction carried out in [7, Section 6] yields, for each singularity component S of the flat orbifold T^7/Γ satisfying condition (i), a fibre bundle[2] $(E_S, \boldsymbol{\varphi}) \to S^2$ with $\{\varphi_b\}_{b \in S^2} = \boldsymbol{\varphi} \in \Omega^3(T^{\mathrm{vert},\vee}E_S)$ a fibre-wise torsion-free G_2-structure. Its classifying map into the homotopy moduli space yields a unique homotopy class of continuous maps

(1.1)

$$\begin{tikzcd}
& \mathcal{G}_2^{\mathrm{tf}}(M)/\!\!/\mathrm{Diff}(M)_0 \arrow[d] & \\
S^2 \arrow[ur, "f_{E_S,\boldsymbol{\varphi}}"] \arrow[r, "f_{E_S}"'] & B\mathrm{Diff}(M)_0 \arrow[r] & B\mathrm{hAut}(M)_0
\end{tikzcd}$$

and Theorem A is now a consequence of the following differential topological result.

Theorem B. *Let M be a simply connected generalised Kummer construction obtained from T^7/Γ as in Theorem A and $\{S\}$ the set of all singularity components satisfying condition (i) and (ii). Then the set $\{[f_{E_S}] \in \pi_2(B\mathrm{hAut}(M)_0)\}$ is linearly independent in $\pi_2(B\mathrm{hAut}(M)) \otimes \mathbb{R}$. In particular,*

$$\mathrm{im}\left[\pi_2(B\mathrm{Diff}(M)_0) \longrightarrow \pi_2(B\mathrm{hAut}(M)_0)\right] \otimes \mathbb{R} \supseteq \mathbb{R}^{\mathrm{N}}.$$

The comparison map $\mathcal{G}_2^{\mathrm{tf}}(M)/\!\!/\mathrm{Diff}(M)_0 \to B\mathrm{Diff}(M)_0$ classifies the functor that sends a G_2-family $(E, \boldsymbol{\varphi})$ to its underlying M-fibre bundle E, while $B\mathrm{Diff}(M)_0 \to B\mathrm{hAut}(M)_0$ classifies the forgetful functor that considers the (isomorphism class of the) M fibre bundle merely as a Serre fibration (up to fibre-homotopy equivalence). Thus, the geometric interpretation of Theorem B is that the underlying topological families $E = \{M_b\}_{b \in S^2}$ are not pair-wise fibre-homotopy inequivalent.

Theorem B can be derived from a robustness principle for orbifold resolutions, applicable to a convenient class of smooth orbifolds of dimension at least 4. To explain it, we need to generalise the notation from Theorem A. Let X be a closed, smooth orbifold in the sense of [20] and $\mathcal{S} \subseteq X$ be a closed, smooth manifold satisfying the following properties:

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[2] In [7], this bundle would be denoted by $E_{M,\{S\}}$.

(i) Each connected component $S \subseteq \mathcal{S}$ has a tubular neighbourhood inside X of the form $\mathrm{Tub}(S) = S \times D^4/\mathbb{Z}_{k_S}$ with $k_S \in \{1, 2\}$.
(ii) $\mathcal{S}$ contains all singular points of X.

We call such a pair $(X, \mathcal{S})$ a *tailor-made* orbifold[3]. Let $\mathsf{N} \subseteq \pi_0(\mathcal{S})$ be the *nice* subset of path components that have a 'homological partner' in the sense that $S \in \mathsf{N}$ if and only if there is a $S' \in \pi_0(\mathcal{S})$ different from S such that the image of their fundamental classes generate the same subvector space $\mathbb{R} \cdot [S] = \mathbb{R} \cdot [S'] \subseteq H_{n-4}(X; \mathbb{R})$.

By condition (ii), we can resolve the orbifold X to a manifold M by replacing the tubular neighbourhood $\mathrm{Tub}(S) \cong S \times D^4/\mathbb{Z}_{k_S}$ of each singularity component by $S \times D\mathcal{O}(\pm k_S)$, where $D\mathcal{O}(k)$ is the disc-bundle of the complex line bundle $\mathcal{O}(k) \to S^2$. Resolving a fixed path component $S \in \pi_0(\mathcal{S})$ in a twisted fashion - that is, using the non-trivial fibre bundle $\mathcal{DO}(k_S) \to S^2$ defined in Subsection 4.1 below instead of $D\mathcal{O}(k_S) \times S^2$, (and the remaining singularities in an untwisted fashion - gives rise to a fibre bundle $M \to E_S \to S^2$, which is classified by a continuous map $f_{E_S} \colon S^2 \to B\mathrm{Diff}(M)_0$ that is unique up to homotopy.

Theorem C. *Let* $(X, \mathcal{S})$ *be a tailor-made orbifold, let* M *be a resolution of* X*, and let* $\mathsf{N} \subseteq \pi_0(\mathcal{S})$ *be the subset of components with a homological partner. Then the set* $\{[f_{E_S}] \in \pi_2(B\mathrm{hAut}(M)_0) : S \in \mathsf{N}\}$ *is linearly independent in* $\pi_2(B\mathrm{hAut}(M)_0) \otimes \mathbb{R}$*. In particular,*

$$\mathrm{im}\left[\pi_2(B\mathrm{Diff}(M)_0) \longrightarrow \pi_2(B\mathrm{hAut}(M)_0)\right] \otimes \mathbb{R} \supseteq \mathbb{R}^{|\mathsf{N}|}.$$

To demonstrate the effectiveness of Theorem C, we provide two immediate consequences:

Corollary D. *For all* $m, n \in \mathbb{N}_0$ *with* $m + n \geq 2$*, we have*

$$\mathrm{im}\left[\pi_2(B\mathrm{Diff}(\mathbb{C}P^{2,\sharp m}\sharp\overline{\mathbb{C}P}^{2,\sharp n})_0) \longrightarrow \pi_2(B\mathrm{hAut}(\mathbb{C}P^{2,\sharp m}\sharp\overline{\mathbb{C}P}^{2,\sharp n})_0)\right] \otimes \mathbb{R} \supseteq \mathbb{R}^{m+n}.$$

Proof. Let $X = S^4$ and $\mathcal{S} = \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ be a set of at least two points. Every point here is regular and has a small tubular neighbourhood of the form D^4, and these neighbourhoods are pairwise disjoint. Resolving the points $\{x_1, \ldots, x_m\}$ by $D\mathcal{O}(1)$ and the points $\{y_1, \ldots, y_n\}$ by $D\mathcal{O}(-1)$ yields the manifold $\mathbb{C}P^{2,\sharp m}\sharp\overline{\mathbb{C}P}^{2,\sharp n}$. Since S^4 is connected, $\mathsf{N} = \pi_0(\mathcal{S})$, and the result follows from Theorem C. □

Remark 1.1. The condition $m + n \geq 2$ cannot be dropped (at least for our construction). If we resolve only a single point $x_1 \in S^4$ with the construction from above, we obtain the bundle $E_{x_1} = \mathcal{DO}(1) \cup_{S^3 \times S^2} D^4 \times S^2$. But it was shown in [13] that the order of the element $[f_{E_x}] \in \pi_2(B\mathrm{Diff}(\mathbb{C}P^2))$ is finite, essentially because one can write down transition functions for E_{x_1} that take values in $\mathrm{PU}(3)$, so $[f_{E_{x_1}}]$ lives in the image of the homomorphism $\pi_2(B\mathrm{PU}(3)) \to \pi_2(B\mathrm{Diff}(\mathbb{C}P^2))$ and the domain $\pi_2(B\mathrm{PU}(3)) = \pi_1(\mathrm{PU}(3)) = \mathbb{Z}_3$ has order 3.

Corollary E. *For the* K3*-surface, we have*

$$\mathrm{im}\left[\pi_2(B\mathrm{Diff}(\mathrm{K3})_0) \to \pi_2(B\mathrm{hAut}(\mathrm{K3})_0)\right] \otimes \mathbb{R} \supseteq \mathbb{R}^{16}.$$

Proof. Recall that the K3-surface can be obtained from the Kummer construction, see [17, Example 7.3.2], as follows: The group $\mathbb{Z}_2 \curvearrowright T^4 = \mathbb{R}^4/\mathbb{Z}^4$ acts diagonally by sending $x + \mathbb{Z}^4$ to $-x + \mathbb{Z}^4$. This action is not free, but has 16 fix-points $\mathcal{S} = \{0, 1/2\}^4 = \mathsf{N}$. A small tubular neighbourhoods around each point is of the form $D^4/\mathbb{Z}_2$, and we obtain the Kummer surface by resolving these singularities with $D\mathcal{O}(-2)$. Theorem C now implies the claim. □

Recently, Baraglia proved [1] that $\pi_2(B\mathrm{Diff}(\mathrm{K3}))$ contains an infinitely generated free abelian subgroup. We believe that our elements form a subgroup of Baraglia's subgroup, which implies that some of these elements are actually in different fibre-homotopy equivalence classes. In addition,

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[3] We remark that the set $\mathcal{S}$ is allowed to more than just singular points

Baraglia's result relies heavily on family Seiberg-Witten invariants, which are not available to higher dimensions at the time of writing.

The reason we study the composition $S^2 \to \mathcal{G}_2^{\mathrm{tf}}(M)/\!\!/\mathrm{Diff}(M)_0 \to B\mathrm{Diff}(M)_0 \to B\mathrm{hAut}(M)$ apart from the intrinsic interest of distinguishing the G_2-families as Serre-fibrations, is that that monoid of homotopy automorphisms $\mathrm{hAut}(M)$ can be effectively studied with the help of rational (or real) homotopy theory, as it allows to compute the rational (or real) homotopy group $\pi_k(\mathrm{hAut}(X)) \otimes \mathbb{K}$ of a finite nilpotent CW complex X in terms of derivations on its minimal model. This idea was originally pioneered by Sullivan [22] and subsequently refined by others; see [18] [6], or [3] for a non-exhaustive list.

The next result generalises Sullivan's observation to the relative situation, namely to the topological monoid $\mathrm{hAut}_A(X)$ of all homotopy equivalences on X that restrict to the identity on A, which we believe to be of independent interest. Recall that a continuous map $f\colon X \to X$ is a homotopy-equivalence relative to A if $f|_A = \mathrm{id}_A$ and there is a continuous map $g\colon X \to X$ with $g|_A = \mathrm{id}_A$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity of X through homotopies H with $H_t|_A = \mathrm{id}_A$ for all $t \in [0, 1]$.

For a commutative differential graded algebra (cdga) B_1 over a field $\mathbb{K}$ of characteristic zero and a differential graded ideal $I \subseteq B_2$, let $\mathrm{Der}_n(B_1, I)$ denote the vector space of all differential $B_1 \to I$ that lower the degree by n. It is easy to see that $\delta\colon \mathrm{Der}_n(B_1, I) \to \mathrm{Der}_{n-1}(B, I)$ given by $\delta(\theta) = d\theta - (-1)^n \theta d$ forms a (homologically graded) differential on $\mathrm{Der}(B_1, I) = \bigoplus \mathrm{Der}_n(B_1, I)$.

Theorem F. *Let $\iota\colon A \hookrightarrow X$ be a pair of finite, nilpotent CW complexes, ΛV_X a Sullivan model for X, and $\mathsf{A}(\iota)\colon \mathsf{A}(X) \twoheadrightarrow \mathsf{A}(A)$ a homomorphism of cdgas modelling the inclusion ι. Then, for all $k \geq 1$, we have*

$$\pi_k(\mathrm{hAut}_A(X), \mathrm{id}) \otimes \mathbb{K} \cong H_k(\mathrm{Der}(\Lambda V_X, \ker \mathsf{A}(\iota)), \delta).$$

Outline of the article: In Section 2, we recall the essential facts and notational conventions of rational and real homotopy theory, and compute the algebraic models of the examples we need later in the article. Section 3 is devoted to proving Theorem F and its generalisation, Theorem 3.13. A central ingredient to give a convenient description of the classifying maps of the fibre bundles $D\mathcal{O}(k) \to \mathcal{D}\mathcal{O}(k) \to S^2$ in terms of derivations on the minimal models of the fibres in Subsection 4.1, which will be used in Subsection 4.2 to prove Theorem C. However, the article is written so that a reader may assume Theorem F as a black-box and immediately jump to the topological application in Section 4 and Section 5. In Section 5, we prove Theorem B apply this theorem to the examples in [16].

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2. Preliminaries on Rational Homotopy Theory

We are going to recall some basic facts of rational homotopy theory that we need in this article. A non-complete list of excellent sources consists of [2], [11], [4], and the original [22]. Although we are going to apply the theory mostly to orbifolds and manifolds, we present it in full generality. It will also be more convenient to work with simplicial sets instead of topological spaces. This is unproblematic because the geometric realisation functor and the singular set functor form a Quillen equivalence $|\cdot|\colon \mathsf{Top} \rightleftarrows \mathsf{sSet}\colon S(\cdot)$. In particular, the two categories have isomorphic homotopy categories. We first give a quick recollection of rational simplicial sets (or rational topological spaces) following [14] and then present the algebraic counterpart mostly following [4].

A connected Kan set is called *nilpotent* if its fundamental group $\pi_1(X)$ is nilpotent and if the action of $\pi_1(X)$ on all higher homotopy groups is nilpotent. A nilpotent Kan set is called *rational* if, for all $n \geq 1$, the power maps $(\,\cdot\,)^n\colon \pi_k(X) \to \pi_k(X)$ are isomorphisms for $k \geq 2$ and bijections for $k = 1$. A simplicial map $\ell_{\mathbb{Q}}\colon X \to X_{\mathbb{Q}}$ between Kan sets is called a *rationalisation* if it satisfies the following universal property: For each rational Kan set and every simplicial map $g\colon X \to Y$ there exists a simplicial map $g_{\mathbb{Q}}$, unique up to homotopy, such that $g_{\mathbb{Q}} \circ \ell_{\mathbb{Q}} \simeq g$. In particular, each simplicial map $f\colon X \to Y$ induces a unique (up to homotopy) simplicial map $f_{\mathbb{Q}}\colon X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ and the rationalisation of a Kan set is unique up to homotopy equivalence. We call two Kan sets X and Y *rational homotopy equivalent* if their rationalisations $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ are (weakly) homotopy equivalent. By Theorem 3B in Chapter II of [14], the rationalisation induces an isomorphism $\pi_k(\ell_{\mathbb{Q}})\colon \pi_k(X) \otimes \mathbb{Q} \to \pi_k(X_{\mathbb{Q}})$, where we use the Maclev-completion for the fundamental group, which agrees with the usual tensor product with $\mathbb{Q}$ if it is abelian, see Chapter I of [14] for details.

On the algebraic side, recall that a *commutative differential graded algebra* (A, d) over a base field $\mathbb{K}$ of characteristic 0 is a (commutative) group object in the category of chain complexes over $\mathbb{K}$. It should be thought of as an abstraction of the de Rham complex of a smooth manifold. A homomorphism of (commutative) differential graded algebras (*dga-homomorphism* for short) is a chain map that respects the multiplicative structure and the unit. Together, they form a category CDGA. We would like to emphasise that we do *not* exclude the possibility that $1 = 0$; this happens if and only if $A = 0$ is the zero-algebra, which is the terminal object in the category CDGA.

To each simplicial set X, we can assign the cdga $\Omega^{\mathbb{K}}_{PL}(X)$ of polynomial differential forms over $\mathbb{K}$, see [4, Definition 2.1] for details. For example, for the (combinatorial) n-simplex, we have

$$\Omega^{\mathbb{K}}_{PL}(\Delta[n]) = \frac{\mathbb{K}[T_0, \ldots, T_n] \otimes \Lambda[dT_0, \ldots, dT_n]}{\langle T_0 + \cdots + T_n = 1, dT_0 + \cdots + dT_n = 0\rangle}$$

with the differential defined as on the smooth differential forms. Usually, we drop $\mathbb{K}$ and the subscript PL from the notation if it does lead to confusion. The algebra over polynomial differential forms gives rise to a contravariant functor that exhibit an adjunction

$$\langle\,\cdot\,\rangle\colon \mathsf{CDGA} \rightleftarrows \mathsf{sSet}^{\mathrm{op}} : \Omega(\,\cdot\,), \tag{2.1}$$

where the left adjoint functor $\langle\,\cdot\,\rangle$ is referred to as the *spatial realisation* and is defined by $\langle B\rangle_n = \mathrm{Hom}_{dga}(B, \Omega(\Delta[n]))$.

The cdga $\Omega(X)$ satisfies a form of the deRham theorem in the sense that $H(\Omega(X), d) \cong H_{\mathrm{simp}}(X; \mathbb{K}) \cong H_{sing}(|X|; \mathbb{K})$, see [4, Theorem 2.2]. To extract (topological) information about X that go beyond its cohomology-ring, we need to replace the rather unwieldy algebra $\Omega(X)$ by more manageable cdgas. To this end, recall that a *quasi-isomorphism* between the cdgas B_1 and B_2 is a dga-homomorphism $\varphi\colon B_1 \to B_2$ that induces an isomorphism between their cohomology groups.

Definition 2.1. An (algebraic) *model* (or $\mathbb{K}$-*model* if we wish to emphasise the underlying field) for a cdga B is a pair (C, m) consisting of another cdga C together with a quasi-isomorphism $m\colon C \to B$. If B_j is modelled by (C_j, m_j) for $j = 1, 2$, we call ψ a *model* for the dga homomorphism $\varphi\colon B_1 \to B_2$ if the following diagram commutes up to homotopy:

$$\begin{tikzcd}
C_1 \arrow[r, "\psi"] \arrow[d, "m_1"'] & C_2 \arrow[d, "m_2"] \\
B_1 \arrow[r, "\varphi"'] & B_2.
\end{tikzcd}$$

We call the models *strict*, if the diagram commutes on the nose. A model for X is a model for $\Omega(X)$ and a model for a continuous map $f\colon X_1 \to X_2$ is a $\mathbb{K}$-model for $\Omega(f)$. If $\mathbb{K}$ is either $\mathbb{Q}$ or $\mathbb{R}$, we refer to them as rational or real models.

We need the definition in this generality only in very few places. Mostly, the underlying cdga of our models will belong to the following subclass, which should be thought of as an algebraic analogue to the subclass of CW-complexes inside the category of topological spaces. The following definition is borrowed from [11].

Definition 2.2. A cdga (B,d) is called a *Sullivan algebra* if there is a graded $\mathbb{K}$-vector space V concentrated in non-negative degrees that has an ascending filtration $V(0) \subseteq V(1) \subseteq \dots$ such that the following three conditions are satisfied:

$$(B,d) = (\Lambda V, d), \qquad \bigcup_{p=0}^{\infty} V(p) = V, \qquad \text{and} \qquad dV(p) \subseteq \Lambda V(p-1),\ dV(0) = \{0\}.$$

A Sullivan algebra $(\Lambda V, d)$ is called *minimal* if $d(V) \subseteq \Lambda^{\geq 2} V$, i.e. if for all $v \in V$ the element $d(v)$ is a linear combination of decomposable elements.

A model (C,m) for a cdga algebra B is called a *Sullivan model* or a *minimal model* if the underlying cdga C is a Sullivan algebra or a minimal algebra, respectively.

We wish to emphasise, as in the case of CW-complexes, that the filtration is not part of the structure.

Example 2.3. Let $(\Lambda V, d)$ be a minimal Sullivan algebra, i.e. $d(V) \subseteq \Lambda^{\geq 2} V$ with $V^0 = 0$. Then the filtration $V(q) = V^{\leq q} := \bigoplus_{s \leq q} V^s$ is a filtration of V. It is easy to see that this filtration turns the differential graded subalgebra $(\Lambda V^{\leq p}, d)$ of $(\Lambda V, d)$ into a (minimal) Sullivan algebra as well. In resemblance to the Postnikov decomposition of a nilpotent topological space, we denote the *p-truncation* $(\Lambda V^{\leq p}, d)$ by $\mathsf{P}_p \Lambda V$.

An important example of a Sullivan algebra that is not minimal the 'algebraic interval' $\Lambda[t, dt]$ that is generated by the graded vector space $V = V^0 \oplus V^1 = \mathbb{R} \cdot t \oplus \mathbb{R} \cdot dt$ and whose differential satisfies the tautological relation $dt = dt$. It comes with two dga homomorphism $\mathrm{ev}_0, \mathrm{ev}_1 \colon \Lambda[t, dt] \to \mathbb{R}$ that satisfy $\mathrm{ev}_j(t) = j$. Note in particular, that there is an isomorphism $\Lambda[t, dt] \cong \Omega(\Delta[1])$ induced by $T_0 \mapsto t$ and $T_1 \mapsto 1 - t$. The algebraic interval allows to define notion of a homotopy for dga-homomorphisms.

Definition 2.4. Two dga-homomorphisms $\varphi_0, \varphi_1 \colon B_1 \to B_2$ are *homotopic* if there is a dga-homomorphism $H \colon B_1 \to \Lambda[t, dt] \otimes B_2$ such that $\mathrm{ev}_j \circ H = \varphi_j$.

The importance of Sullivan algebras ΛV is that being homotopic induces an equivalence relation on $\mathrm{Hom}_{dga}(\Lambda V, B)$, [11, Proposition 12.7] and composition with a quasi-isomorphism $\varphi \colon B_1 \to B_2$ induces a bijection on the set of homotopy classes $[\Lambda V, B_1] \xrightarrow{\cong} [\Lambda V, B_2]$.

By [11, Theorem 14.12] every cdga B with $H^0(B) = \mathbb{K}$ has a minimal model and for two minimal models $m_j \colon C_j \to B$ there is an isomorphism $\Phi \colon C_1 \to C_2$ such that m_1 and $m_2 \circ \Phi$ are homotopic. For this reason, we will denote the minimal model of a simplicial set X (or topological space) with M_X. Moreover, each dga-homomorphism $\varphi \colon B_1 \to B_2$ gives rise to a homomorphism $\mathsf{M}(\varphi) \colon \mathsf{M}_1 \to \mathsf{M}_2$ that is unique in its homotopy class.

It was proved by Bousfield and Gugenheim that the category CDGA carries a closed model structure with quasi-isomorphisms as weak-equivalences and surjective dga-homomorphism as fibrations, see [4, Theorem 4.3]. In this model structure, Sullivan algebras are cofibrant objects, which can be derived from [4, 4.5 Closure Properties], but see [2, Theorem 8.11] for an explicit proof.

The adjunction (2.1) gives rise to a Quillen adjunction, but it fails to induce an equivalence between their homotopy categories (even if one only considers rational simplicial sets).However, if $\mathbb{K} = \mathbb{Q}$ and if $\mathrm{fin}_{\mathbb{Q},\mathrm{nil}}\text{-}\mathrm{Ho}(\mathsf{sSet})$ denotes the full subcategory of the homotopy category of sSet whose objects are connected, nilpotent, Kan complexes of finite $\mathbb{Q}$-type (i.e. $H_i(X;\mathbb{Q})$ is finite dimensional for all $i \geq 0$) and if $\mathrm{fin}_{\mathbb{Q}}\text{-}\mathrm{Ho}(\mathsf{CDGA})$ denotes the full subcategory of $\mathrm{Ho}(\mathsf{CDGA})$ whose objects are homologically connected (i.e. $H^0(C,d) = \mathbb{Q}$) cdgas of finite $\mathbb{Q}$-type, then , then Bousfield and Gugenheim proved that the spacial realisation and the minimal model induce an equivalence of categories

$$\langle \cdot \rangle \colon \mathrm{fin}_{\mathbb{Q}}\text{-}\mathrm{Ho}(\mathsf{CDGA}) \overset{\cong}{\rightleftarrows} \mathrm{fin}_{\mathbb{Q},\mathrm{nil}}\text{-}\mathrm{Ho}(\mathsf{sSet})^{\mathrm{op}} : \mathsf{M},$$

see [4, Theorem 9.4].

Unpacking the notion of homotopy categories leads to the following explicit consequences.

Proposition 2.5. *If X_j with $j = 0, 1$ are nilpotent Kan sets (or CW-complexes) whose homology groups $H_*(X_j; \mathbb{Q})$ is finite dimensional in every degree, then the following statements hold true:*

(1) X_0 and X_1 are rational homotopy equivalent if and only if their minimal models are isomorphic.

(2) If, in addition, X_0 and X_1 are rational, then the set of homotopy classes agrees with the set of homotopy classes between their Sullivan models:

$$[X, Y] \cong [\Lambda V_Y, \Lambda V_X]$$

(3) If M_X is the minimal model, then the map $\ell\colon X \mapsto \langle \mathsf{M}_X \rangle$, which should be thought of as the unit of the adjunction (2.1), is a rationalisation.

The minimal model of a nilpotent Kan set (over each field $\mathbb{K}$ of characteristic zero) is closely connected to its homotopy group.

Proposition 2.6 (Theorem 11.3 in [4])**.** *If X a nilpotent Kan complex of finite $\mathbb{Q}$-type and $\mathsf{M}_X = \Lambda V_X$ is minimal model, then there is a natural bijection $\pi_k(X) \otimes \mathbb{K} \cong \mathrm{Hom}(V^k_X, \mathbb{K})$, which is a homomorphism whenever $\pi_k(X)$ is abelian.*

Since we will mostly deal with manifolds and orbifolds, it is more convenient to work out the examples over the real numbers using the de Rham complex of smooth differential forms. The next result was stated in [11, p.135ff] for smooth manifolds, but the proof given carries over to orbifolds.

Proposition 2.7. *Let M be a smooth manifold or a smooth orbifold in the sense of [24], then $\Omega^{\mathbb{R}}_{PL}(M)$ and $\Omega_{dR}(M)$ are quasi-isomorphic.*

In particular, if M is closed (or more general if it is $\mathbb{Q}$-finite), then $\mathsf{M}^{\mathbb{R}}_M = \mathsf{M}^{\mathbb{Q}}_M \otimes \mathbb{R}$ and we can derive the real minimal model from the de Rham algebra. We use this insight to close this section with a collection of examples we need in forthcoming sections.

Example 2.8. If n is even, then the minimal model of the n-dimensional sphere S^n is given by $\mathsf{M}_{S^n} = \Lambda[a_n, b_{2n-1} \,|\, db_{2n-1} = a_n^2]$. The model map $m\colon \mathsf{M}_{S^n} \to \Omega(S^n)$ sends a_n to 'the' volume form vol_{S^n} and b_{2n-1} to zero.

If n is odd, then the minimal model of the n-dimensional sphere S^n, or more generally, for a lens space $S^n/\mathbb{Z}_k$, is given by $\mathsf{M}_{S^n/\mathbb{Z}_k} = \Lambda[a_n]$ with $\deg(a_n) = n$ and vanishing differential. The model map $m\colon \mathsf{M}_{S^n/\mathbb{Z}_k} \to \Omega(S^n/\mathbb{Z}_k)$ sends a_n to 'the' volume form $\mathrm{vol}_{S^n/\mathbb{Z}_k}$.

Example 2.9. For all $k \in \mathbb{Z}$, the disc bundles of the complex line bundles $\mathcal{O}(k) \to \mathbb{C}P^1$ can be constructed from Hopf-fibration via a Borel construction:

$$D\mathcal{O}(k) = S^3 \times_{S^1,(\,\cdot\,)^k} D^2 = (S^3 \times D^2)/\sim \qquad (p, \lambda) \sim (\mathrm{e}^{\mathrm{i}\theta} p, \mathrm{e}^{k\mathrm{i}\theta}\lambda).$$

They are manifolds with boundary $S^3/\mathbb{Z}_k$ and they are all homotopy equivalent to $\mathbb{C}P^1$. Thus, their minimal models are given by $\mathsf{M}_{D\mathcal{O}(k)} = \Lambda[a_2, b_3 \,|\, db_3 = a_2^2]$.

To model the boundary inclusion $\iota\colon \partial D\mathcal{O}(k) \hookrightarrow D\mathcal{O}(k)$, we choose the following model map $m\colon \mathsf{M}_{D\mathcal{O}(k)} \to \Omega_{dR}(D\mathcal{O}(k))$: The generator a_2 gets mapped to a Thom form τ, which is a compactly supported 2-form on $D\mathcal{O}(k) \setminus S^3/\mathbb{Z}_k$ whose integral over each fibre $D^2 \hookrightarrow D\mathcal{O}(k)$ is 1. By the Thom isomorphism theorem, we have $\tau^2 = \mathrm{e}(\mathcal{O}(k))\mathrm{vol}_{D\mathcal{O}(k)} = k \cdot \mathrm{vol}_{D\mathcal{O}(k)}$, where is a four-form on $D\mathcal{O}(k)$ whose support does not intersect the boundary and that integrates to 1 over $D\mathcal{O}(k)$. Let $\tilde{\omega} \in \Omega^3(D\mathcal{O}(k)$ be a primitive of $\mathrm{vol}_{D\mathcal{O}(k)}$ and $\eta \in \Omega^2(S^3/\mathbb{Z}_k)$ such that

$$d\eta = \omega|_{S^3/\mathbb{Z}_k} - \int_{S^3/\mathbb{Z}_k} \omega \cdot \mathrm{vol}_{S^3/\mathbb{Z}_k} = \omega|_{S^3/\mathbb{Z}_k} - \mathrm{vol}_{S^3/\mathbb{Z}_k},$$

with $\mathrm{vol}_{S^3/\mathbb{Z}_k}$ the generator from the previous example. Let, moreover, $\chi\colon D\mathcal{O}(k)\to[0,1]$ be a cut-off function that is identically 1 near the boundary and whose support is disjoint from the one of $\mathrm{vol}_{D\mathcal{O}(k)}$. We now define $m(b_3)=k\omega:=k(\tilde{\omega}-d(\chi\eta))$.

With this model map at hand, it is clear now that the dga-homomorphism $\Omega(\iota)$ is modelled by the maps

$$\mathsf{M}_{D\mathcal{O}(k)}=\Lambda[a_2,b_3]\to\mathsf{M}_{\partial D\mathcal{O}(k)}=\Lambda[\beta_3],\qquad\qquad b_3\mapsto k\cdot\beta_3.$$

Remark 2.10. Since the composition $S^3\hookrightarrow\mathcal{O}(-1)\to\mathbb{C}P^1$ is the Hopf-fibration, the previous example also shows that the Hopf fibration has the dga-homomorphism $\mathsf{M}_{S^3}=\Lambda[a_2,b_3]\to\Lambda[\beta_3]=\mathsf{M}_{S^2}$ that is given by $b_3\mapsto\beta_3$ as a rational model.

Example 2.11. The minimal model of a complex projective space is given by the cdga $\mathsf{M}_{\mathbb{C}P^n}=\Lambda[a_2,b_{2n+1}\,|\,db_{2n+1}=a_2^{n+1}]$. The model map $m\colon\mathsf{M}_{\mathbb{C}P^n}\to\Omega(\mathbb{C}P^n)$ sends a_2 to $\omega_{FS}/\sqrt{2}$, the Kähler form of the Fubini-Study metric and b_{2n+1} to zero.

The next example discussion rational models of (homotopy) push outs of simplicial sets (or CW complexes). Details can be found in [11, Section 13.1].

Example 2.12. Let $B_1\xrightarrow{\varphi_1}B_0\xleftarrow{\varphi_2}B_2$ be dga-homomorphisms between two cdgas. Its *fibre product* $B_1\oplus_{B_0}B_2$ is the cdga

$$B_1\oplus_{B_0}B_2=\{(x,y)\,:\,\varphi_1(x)=\varphi_2(y)\}\subseteq B_1\oplus B_2,$$

where the differential d acts component-wise.

Let X_0, X_1, X_2 be simplicial sets and $\iota_j\colon X_0\to X_j$ with one map being a cofibration. If one of the models $\mathsf{A}(\iota_j)\colon\mathsf{A}(X_j)\to\mathsf{A}(X_0)$ is surjective, then the fibre product serves $\mathsf{A}(X_1)\oplus_{\mathsf{A}(X_0)}\mathsf{A}(X_2)$ is a rational model for the (homotopy) push out Y of $X_1\xleftarrow{\iota_1}X_0\xrightarrow{\iota_2}X_2$.

Besides allowing for a calculation of real homotopy groups out of the de Rham complex, we can use the minimal model of a compact manifold M with boundary ∂M to get information about the rational homotopy groups of its homotopy automorphisms (relative to the boundary) using Theorem F. As a demonstration, we present these calculations for the manifolds $D\mathcal{O}(k)$ with boundary $S^3/\mathbb{Z}_k$ and our results to the real homotopy groups of the topological monoid $\mathrm{hAut}(D\mathcal{O}(k))$ of 'absolute' homotopy equivalences.

Example 2.13. We have seen in Example 2.9 that the minimal models of $D\mathcal{O}(k)$ and its boundary $S^3/\mathbb{Z}_k$ are given by $\mathsf{M}_{D\mathcal{O}(k)}=\Lambda[a_2,b_3\,|\,db_3=a_2^2]$ and $\mathsf{M}_{S^3/\mathbb{Z}_k}=\Lambda[\beta_3]$. Moreover, it was shown that the real model for the boundary inclusion is modelled by the dga-homomorphism $\mathsf{M}(\iota)\colon\mathsf{M}_{D\mathcal{O}(k)}\to\mathsf{M}_{S^3/\mathbb{Z}_k}$ that send a_2 to 0 and b_3 to β_3. In particular, $\ker\mathsf{M}(\iota)=\bigoplus_{n\neq 0,3}\mathsf{M}^n_{D\mathcal{O}(k)}$.

Since (graded) derivations from a free algebra are completely determined by the image of its generating vector space, we observe that

$$\mathrm{Der}_n(\mathsf{M}_{D\mathcal{O}(k)},\mathsf{M}_{D\mathcal{O}(k)})=\begin{cases}\mathbb{R}, & \text{if } n\leq 3,\\ 0, & \text{if } n\geq 4,\end{cases}\quad\text{while}\quad\mathrm{Der}_n(\mathsf{M}_{D\mathcal{O}(k)},\ker\mathsf{M}(\iota))=\begin{cases}\mathbb{R}, & \text{if } n=1,\\ 0, & \text{if } n\geq 2.\end{cases}$$

with $\mathrm{Der}(\mathsf{M}_{D\mathcal{O}(k)},\mathsf{M}_{D\mathcal{O}(k)})$ generated by the linear maps $a_2\otimes b_3^\vee$, $1\otimes a_2^\vee$, and $1\otimes b_3^\vee$ of degree 1, 2, and 3. Observe that the latter two linear maps take non-zero values in $\mathsf{M}^0_{D\mathcal{O}(k)}=\mathbb{R}\cdot 1$, which does not lie in the differential ideal $\ker\mathsf{M}(\iota)$.

A straight-forward computation shows that

$$\delta_1(a_2\otimes b_3^\vee)=0,\quad\delta_2(1\otimes a_2^\vee)=2a_2\otimes b_3^\vee,\quad\text{and}\quad\delta_3(1\otimes b_3^\vee)=0,$$

and Theorem F now implies that

$$\pi_n(\mathrm{hAut}(D\mathcal{O}(k)),\mathrm{id})\otimes\mathbb{R}\cong H_n(\mathrm{Der}(\mathsf{M}_{D\mathcal{O}(k)},\mathsf{M}_{D\mathcal{O}(k)}),\delta)=\begin{cases}\mathbb{R}, & \text{if } n=3,\\ 0, & \text{if } n\neq 3,\end{cases}$$

while

$$\pi_n(\mathrm{hAut}_\partial(D\mathcal{O}(k)),\mathrm{id})\otimes\mathbb{R}\cong H_n(\mathrm{Der}(\mathsf{M}_{D\mathcal{O}(k)},\ker\mathsf{M}(\iota)),\delta)=\begin{cases}\mathbb{R}, & \text{if } n=1,\\ 0, & \text{if } n\neq 1.\end{cases}$$

3. Algebraic models for homotopy automorphisms

In this section, we are going to study mapping spaces using rational homotopy theory. The main goal of this section is to give a detailed proof of Theorem F and its generalisation, Theorem 3.13 below, which state that the rational homotopy groups of a (relative) mapping space $\mathrm{C}_f(X,Y)$ can be computed in terms of derivations between the rational models of X and Y.

The presentation closely follows the philosophy of [4] by mostly relying on homotopical algebra. In fact, we are going to generalise Section 5.8 in loc. cit. from augmentations to general surjective dga-homomorphisms. We first explore the relation between the topological and simplicial mapping spaces to their algebraic counterparts and, in a second step, relate the homotopy groups of the algebraic mapping spaces to appropriate homology groups of derivations.

Topological and Simplicial Mapping Spaces. Although the category of simplicial sets and topological spaces are Quillen equivalent (via the geometric realisation functor $|\cdot|$ and the singular set functor $S(\cdot)$), it is worthwhile to recall their precise relation.

For two (compactly generated Hausdorff) spaces X and Y, let $\mathrm{C}(X,Y)$ be the space of all continuous maps equipped with the kellification of the compact-open topology; that is a subset of $\mathrm{C}(X,Y)$ is closed if and only if $A\cap C$ is closed with respect to the compact open topology for every compact subspace $C\subseteq\mathrm{C}(X,Y)$. In this way, mapping spaces exhibit an exponential law $\mathrm{C}(X\times Y,Z)\cong\mathrm{C}(X,\mathrm{C}(Y,Z))$. If $\iota\colon A\hookrightarrow X$ is a cofibration, then $|_A=(\cdot)\circ\iota_A\colon\mathrm{C}(X,Y)\to\mathrm{C}(A,Y)$ is a Serre fibration and the fibre of $f\colon A\to Y$ is denoted by $\mathrm{C}_f(X,Y)$ or $\mathrm{C}_{A,f}(X,Y)$ if we wish to emphasise the subspace as well. If $X=Y$ and $f=\mathrm{id}$, we may simply write $\mathrm{C}_A(X)$.

For two simplicial sets X and Y, the simplicial analogue of a mapping space is the simplicial set $\mathrm{map}(X,Y)$ whose set of n-simplicies is given by the set

$$\mathrm{map}_n(X,Y)=\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n]\times X,Y)$$

and the structure maps are precomposition with $\phi\times\mathrm{id}_X$ where $\phi\colon\Delta[m]\to\Delta[n]$ is a morphism in the simplex category.

It is known that $\mathrm{map}(X,Y)$ is a Kan set if the target Y is a Kan set, see [10, Prop 1.17], and that $\mathrm{map}(X,Y)$ is the internal right-adjoint functor to the product functor [12, Proposition I.5.1], which means that, for all simplicial set Z, there exists a natural bijection, the 'simplicial exponential law',

$$\mathrm{Hom}_{\mathsf{sSet}}(Z\times X,Y)\cong\mathrm{Hom}_{\mathsf{sSet}}\left(Z,\mathrm{map}(X,Y)\right).$$

If X,Y, and Z are simplicial spaces, then the composition of simplicial maps extends to a simplicial map

$$\mathrm{map}(X,Y)\times\mathrm{map}(Y,Z)\to\mathrm{map}(X,Z).$$

Explicitly, the composition $f\circ g$ of $g\in\mathrm{map}_n(X,Y)=\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n]\times X,Y)$ with $f\in\mathrm{map}_n(Y,Z)=\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n]\times Y,Z)$ is given by

$$\Delta[n]\times X\xrightarrow{(\mathrm{id}_{\Delta[n]},g)}\Delta[n]\times Y\xrightarrow{\quad f\quad}Z.$$

For every topological space X, the counit $\mathrm{ev}\colon|S(X)|\to X$ is a weak homotopy equivalence, where $S(X)$ denotes the singular set of X, see [10, p. 207]. Furthermore, observe that the graph construction yields a simplicial map

$$S(\mathrm{C}(X,Y))\to\mathrm{map}(S(X),S(Y))$$

that sends an element $f\colon \Delta^n \to \mathrm{C}(X,Y)$ of $S_n(\mathrm{C}(X,Y))$ with adjoint map $\mathrm{Ad}(f)\colon X \times \Delta^n \to X$ to the simplicial map $S(X) \times \Delta[n] \to S(X)$ that is given by

$$S_m(X) \times \Delta[n]_m \ni (\sigma, \varphi\colon \Delta^m \to \Delta^n) \mapsto (\Delta^m \xrightarrow{\sigma,\varphi} X \times \Delta^n \xrightarrow{\mathrm{Ad}(f)} X) \in S_m(X).$$

Since the two simplicial sets are Kan, it is easy to see using the combinatorial description of their homotopy groups that this map is a weak homotopy equivalence and induces the canonical identification[4]

$$\pi_n(\mathrm{C}(X,Y), f) \cong \pi_0(\mathrm{C}_{\{*\}\times X, f}(S^n \times X, X)).$$

We thus have a zig-zag of weak homotopy equivalences

$$|\mathrm{map}(S(X), S(Y))| \xleftarrow{\simeq} |S(\mathrm{C}(X,Y))| \xrightarrow{\simeq} \mathrm{C}(X,Y), \tag{3.1}$$

and so we are allowed to work with $S(\mathrm{C}(X,Y))$, if we desire to apply simplicial methods. The geometric realisation turns a Kan-fibration into a Serre fibration and the singular set turn a Serre fibration into a Kan fibration, see [12, Theorem 10.10] for the latter. Since this zig-zag (3.1) is natural in both components, we also get a zig-zag for the relative mapping spaces:

$$|\mathrm{map}_{Sf}(S(X), S(Y))| \xleftarrow{\simeq} |S(\mathrm{C}_f(X,Y))| \xrightarrow{\simeq} \mathrm{C}_f(X,Y). \tag{3.2}$$

Algebraic Mapping Spaces. Following Bousfield and Gugenheim [4], we construct the counterpart of mapping spaces in the category of commutative differential graded algebras CDGA over a field $\mathbb{K}$ of characteristic zero.

Definition 3.1. For two cdgas B_0 and B_1, the simplicial set $\mathrm{map}(B_1, B_2)$ is defined as follows: The set of all n-simplices is

$$\mathrm{map}_n(B_1, B_2) := \mathrm{Hom}_{\mathsf{sSet}}(B_1, \Omega(\Delta[n]) \otimes B_2)$$

and face and degeneracy maps are given by composition with $\Omega(\phi) \otimes \mathrm{id}_{B_2}$ where $\phi\colon \Delta[m] \to \Delta[n]$ is a morphism in the simplex category.

Note that this is a generalisation of the spacial realisation functor: $\langle B_1 \rangle = \mathrm{map}(B_1, \mathbb{K})$.

We remind the reader that a cofibration in CDGA is a dga-homomorphism $C_1 \to C_2$ that has the left lifting property with respect to all surjective quasi-isomorphism, i.e. for each commutative outer square exists a dashed filler:

$$\begin{tikzcd}
C_1 \arrow[r] \arrow[d, hook, "\iota"'] & B_1 \arrow[d, two heads, "\simeq"] \\
C_2 \arrow[r] \arrow[ur, dashed] & B_2.
\end{tikzcd}$$

If $C_1 = \mathbb{K} \cdot 1$ and ι is the unit map, then C_2 is called cofibrant, and Sullivan algebras are examples of such.

We will derive many of our results from the following 'fundamental result'.

Theorem 3.2. *[4, Theorem 5.3] Let $\iota\colon C_1 \to C_2$ be a (dga-)cofibration and let $p\colon B \to T$ be a surjective dga-homomorphism. Then the natural simplicial map*

$$\iota * p = ((\,\cdot\,) \circ \iota, p \circ (\,\cdot\,))\colon \mathrm{map}(C_2, B) \to \mathrm{map}(C_1, B) \times_{\mathrm{map}(C_1,T)} \mathrm{map}(C_2, T)$$

is a Kan fibration. If, in addition, ι or p is a quasi-isomorphism, the $\iota * p$ is a weak-equivalence.

We will mostly use the following special case where $C_1 = \mathbb{K} \cdot 1$.

Corollary 3.3. *If C is a cofibrant cdga, e.g. a Sullivan algebra, and $p\colon B \to T$ a surjective dga homomorphism, then the map $p \circ (\,\cdot\,)\colon \mathrm{map}(C, B) \to \mathrm{map}(C, T)$ is a Kan fibration. If p is additional a quasi-isomorphism, then $p \circ (\,\cdot\,)$ is a weak equivalence.*

---

[4] Of course, this can also be deduced from the exponential law.

Corollary 3.4. *Let* B *and* C *be two cdgas with* C *cofibrant. Then* $\operatorname{map}(C, B)$ *is a Kan set. In particular, if* X *a simplicial set, then the simplicial sets* $\operatorname{map}(X, \langle C\rangle)$ *and* $\operatorname{map}(C, \Omega X)$ *are Kan sets.*

Proof. By definition, $\iota\colon \mathbb{K} \hookrightarrow C$ is a cofibration, so by Theorem 3.2 the restriction $\operatorname{map}(C, \Omega X) \to \operatorname{map}(\mathbb{K}, B) = \{1_B\}$ is Kan fibration; hence $\operatorname{map}(C, B)$ is a Kan set. It follows, in particular, that $\langle C\rangle = \operatorname{map}(C, \mathbb{K})$ is Kan, so the simplicial mapping space $\operatorname{map}(X, \langle C\rangle)$ is Kan, too. ☐

The next result should be thought of as a space-level version of Whitehead's theorem in the algebraic set-up.

Proposition 3.5. *[4, Proposition 5.7] Let* C *be a cofibrant cdga and let* $f\colon B_1 \to B_2$ *be a (not necessarily surjective) quasi-isomorphism. Then post composition with* f *induces a weak equivalence* $f \circ (\,\cdot\,)\colon \operatorname{map}(C, B_1) \to \operatorname{map}(C, B_2)$.

We now turn our attention to relative mapping sets that should be thought of as the algebraic counterparts to $\mathrm{C}_f(X, Y)$.

Definition 3.6. Let $B_2 \twoheadrightarrow T$ be a surjective dga-homomorphism and let $\varphi\colon B_1 \to T$ be a fixed dga-homomorphism. Define the φ-relative mapping set $\operatorname{map}^{T,\varphi}(B_1, B_2)$ as the following pullback

$$\begin{tikzcd}
\operatorname{map}^{T,\varphi}(B_1, B_2) \arrow[r] \arrow[d] & \operatorname{map}(B_1, B_2) \arrow[d, "p\circ(\,\cdot\,)"] \\
* \arrow[r, "\varphi"'] & \operatorname{map}(B_1, T).
\end{tikzcd}$$

If it does not lead to confusion, we will either drop T or φ from the decoration. Note that, as $p \circ (\,\cdot\,)$ is a Kan fibration, the weak homotopy type of $\operatorname{map}^{T,\varphi}(B_1, B_2)$ only depends on the homotopy class of $\varphi\colon B_1 \to T$.

The next results essentially says that we can replace cdgas with their rational models without changing the weak homotopy type of their relative mapping sets.

Lemma 3.7. *Let* $H\colon B_1 \to \Lambda[t, dt] \otimes T_2$ *be a homotopy that makes the diagram, in which the horizontal arrows are quasi-isomorphism,*

$$\begin{tikzcd}
B_1 \arrow[r, "F"] \arrow[d, two heads, "p_1"'] & B_2 \arrow[d, two heads, "p_2"] \\
T_1 \arrow[r, "f"'] & T_2
\end{tikzcd}$$

commutative up to homotopy. Let C *be a cofibrant cdga,* $\Phi\colon C \to B_1$ *a dga-homomorphism, and set* $\varphi = p_1 \circ \Phi$. *Then the homotopy* H *induces a weak equivalence* $\operatorname{map}^{T_1,\varphi}(C, B_1) \to \operatorname{map}^{T_2, p_2 F \Phi}(C, B_2)$.

Proof. By definition of the algebraic mapping space, the homotopy H defines 1-simplex inside $\operatorname{map}(B_1, T_2)$, which can be represented by a simplicial map $H\colon \Delta[1] \to \operatorname{map}(B_1, T_2)$. Composition with H yields a homotopy $\mathrm{H}\colon \Delta[1] \times \operatorname{map}(C, B_1) \to \operatorname{map}(C, T_2)$, which makes the following diagram of Kan fibrations commutative up to homotopy

$$\begin{tikzcd}
\operatorname{map}(C, B_1) \arrow[r, "\Phi\circ(\,\cdot\,)"] \arrow[d, "p_1\circ(\,\cdot\,)"'] & \operatorname{map}(C, B_2) \arrow[d, "p_2\circ(\,\cdot\,)"] \\
\operatorname{map}(C, T_1) \arrow[r, "f\circ(\,\cdot\,)"'] & \operatorname{map}(C, T_2).
\end{tikzcd}$$

Since right the vertical is a Kan-fibration, we can lift the homotopy $\mathrm{H}\colon \Delta[1] \times \operatorname{map}(C, B_1) \to \operatorname{map}(C, T_2)$ to a homotopy $\mathscr{H}\colon \Delta[1] \times \operatorname{map}(C, B_1) \to \operatorname{map}(C, T_2)$ extending Φ on $\{1\} \times \operatorname{map}(C, B_1))$.

If G is the restriction of $\mathscr{H}$ to $\{0\}\times \mathrm{map}(C,B_1)$, then G is a map of Kan fibrations (over f), so it induces a map of fibres $G\colon \mathrm{map}^{T_1,\varphi}(C,B_1)\to \mathrm{map}^{T_2,p_2G\Phi}(C,T_2)$.

Since H is a path inside the Kan set $\mathrm{map}(C,T_2)$ between $p_2G(\Phi)=f\varphi$ and $p_2F\Phi$, it gives rise to a weak equivalence $\hat{H}\colon \mathrm{map}^{T_2,p_2G(\Phi)}(C,B_2)\to \mathrm{map}^{T_2,pF\Phi}(C,B_2)$.

Since F and f are quasi-isomorphism, $F\circ(\,\cdot\,)$ and $f\circ(\,\cdot\,)$ are weak equivalences by Proposition 3.5, so G is a weak equivalence and, by the Five lemma, its restriction to the fibre $\mathrm{map}^{T_1,\varphi}(C,B_1)$ is a weak equivalence, too. Thus, the composition of $\hat{H}\circ G\colon \mathrm{map}^{T_1,\varphi}(C,B_1)\to \mathrm{map}^{T_2,pF\Phi}(C,B_2)$ is a weak equivalence we were looking for. □

The natural bijection $\mathrm{Hom}_{dga}(B,\Omega X)\cong \mathrm{Hom}_{\mathsf{sSet}}(X,\langle B\rangle)$ of the adjunction (2.1) can be used to define a map of simplicial sets

$$\mu\colon \mathrm{map}(B,\Omega X)\to \mathrm{map}(X,\langle B\rangle).$$

On the level of n-simplices, the natural map given by (external) multiplication

$$\mu_n\colon \Omega(X)\otimes\Omega(\Delta[n])\to\Omega(X\times\Delta[n]),\qquad (x,y)\mapsto \Omega(\mathrm{pr}_1)(x)\cdot\Omega(\mathrm{pr}_2)(y),$$

which is, in fact, a quasi-isomorphism of cdgas by the Künneth formula. Together with the following adjunctions, the external product yield a sequence of (natural) maps

$$\begin{aligned}\mathrm{map}_n(B,\Omega(X)) &=\mathrm{Hom}_{dga}(B,\Omega(X)\otimes\Omega(\Delta[n]))\xrightarrow{\mu_n\circ(\,\cdot\,)}\mathrm{Hom}_{dga}(B,\Omega(X\times\Delta[n]))\\ &\cong\mathrm{Hom}_{\mathsf{sSet}}(X\times\Delta[n],\langle B\rangle)\\ &=\mathrm{map}_n(X,\langle B\rangle),\end{aligned}$$

and naturality of the adjunction implies that these maps form a simplicial map μ.

Proposition 3.8. *If C is a cofibrant cdga, then the natural map $\mu\colon \mathrm{map}(C,\Omega(X))\to\mathrm{map}(X,\langle C\rangle)$ is a weak homotopy equivalence.*

Proof. Using the combinatorial description of homotopy groups for Kan sets, we see that $\pi_n(\mu)$ decomposes as follows:

$$\begin{aligned}\pi_n(\mathrm{map}(C,\Omega(X)),\varphi)=\pi_0(\mathrm{map}^{\Omega(X),\varphi}(C,\Omega(S^n)\otimes\Omega(X)))&\to\pi_0(\mathrm{map}^{\Omega(X),\varphi}(C,\Omega(S^n\times X))\\ &\to\pi_0(\mathrm{map}_{X,\varphi}(S^n\times X,\langle C\rangle))\\ &=\pi_n(\mathrm{map}(X,\langle C\rangle),\varphi).\end{aligned}$$

Since the (external) tensor product $\Omega(S^n)\otimes\Omega(X)\to\Omega(S^n\times X)$ is a quasi-isomorphism, the first map in the composition is an isomorphism by Theorem 3.2.

While the two simplicial sets $\mathrm{map}^{\Omega(X),\varphi}(C,\Omega(S^n\times X))$ and $\mathrm{map}_{X,\varphi}(S^n\times X,\langle C\rangle)$ have the same 0-simplices, the latter has more 1-simplices. Hence, we need to show that the second map is injective, which we accomplish by generalising the proof of Lemma 8.43 in [2] to the relative set-up. Let $H\colon\Delta[1]\times S^n\times X\to\langle C\rangle$ be a homotopy between ψ_0 and ψ_1 that restricts to $\varphi\circ\mathrm{pr}_X$ on $\Delta[1]\times *\times X$. Under the adjunction of $\langle\,\cdot\,\rangle$ and $\Omega(\,\cdot\,)$, the homotopy H corresponds to a dga-homomorphism $\mathrm{Ad}(H)\colon C\to\Omega(\Delta[1]\times S^n\times X)$ whose composition with the face maps ∂^0 and ∂^1 yields ψ_0 and ψ_1, respectively.

Since face maps $\partial^j\colon\Omega(\Delta[1]\times S^n\times X)\to\Omega(S^n\times X)$ and the degeneracy map $\sigma_0\colon\Omega(S^n\times X)\to\Omega(\Delta[1]\times S^n\times X)$ are quasi-isomorphisms, we obtain a zig-zag of weak equivalences

$$\mathrm{map}^{\Omega(X),\varphi}(C,S^n\times X)\underset{\sigma_0}{\overset{\partial^0}{\leftrightarrows}}\mathrm{map}^{\Omega(\Delta[1]\times X),\varphi\circ\mathrm{pr}_X}(C,\Omega(\Delta[1]\times S^n\times X))\underset{\sigma_0}{\overset{\partial^1}{\rightleftarrows}}\mathrm{map}^{\Omega(X),\varphi}(C,S^n\times X).$$

Since $\partial^j\sigma_0 = \mathrm{id}$ and ∂^j are weak equivalences, we deduce that $\pi_0(\sigma_0)$ is the inverse of $\pi_0(\partial^j)$. Since $\mathrm{Ad}(H)$ is a 0-simplex in $\mathrm{map}^{\Omega(\Delta[1]\times X),\varphi\circ\mathrm{pr}_X}(C, \Omega(\Delta[1] \times S^n \times X))$, we deduce that

$$\begin{aligned}[\psi_0] &= \pi_0(\partial^0)([\mathrm{Ad}(H)]) = \pi_0(\partial^0)(\pi_0(\sigma_0)([\psi_0])) = \pi_0(\partial^1)(\pi_0(\sigma_0)([\psi_0])) \\ &= \pi_0(\partial^0)([\mathrm{Ad}(H)]) = [\psi_1] \in \pi_0(\mathrm{map}^{\Omega(X),\varphi}(C, S^n \times X)),\end{aligned}$$

so injectivity is proved. □

Because μ is natural, we immediately conclude the following consequnce.

Corollary 3.9. *If $A \hookrightarrow X$ is an inclusion of simplicial sets and $\varphi\colon C \to \Omega(A)$ a fixed dga homomorphism with C a cofibrant cdga, then the natural map μ induces a weak equivalence*

$$\mathrm{map}^{\Omega(A),\varphi}(C, \Omega(X)) \to \mathrm{map}_{A,\varphi}(X, \langle C\rangle).$$

Now, let K be a connected, nilpotent Kan set of finite $\mathbb{Q}$-type and $\Lambda V_K \to \Omega(K)$ a Sullivan model for $\Omega(K)$. Let further $\mathsf{A}(\iota)\colon \mathsf{A}(X) \twoheadrightarrow \mathsf{A}(A)$ a surjective model for the inclusion $\iota\colon A \hookrightarrow X$. Let $\eta\colon K \to \langle\Lambda V_K\rangle$ be a lift of the unit map $K \to \langle\Omega(K)\rangle$, which is unique up to homotopy and induces an isomorphism on rational homotopy groups, see [4, Thm 11.2]. For $f\colon A \to K$, denote the composition $\eta \circ f$ with φ, which can be also interpreted as a dga homomorphism $\varphi\colon \Lambda V_K \to \Omega(A)$. Let further $\psi\colon \Lambda V_K \to \mathsf{A}(A)$ be a lift of φ under the model map $\mathsf{A}(A) \to \Omega(A)$, which exists and is unique up to homotopy by Proposition 3.5. Then, by Lemma 3.7 there is a zig-zag of weak homotopy equivalences

$$\mathrm{map}_f(X, K) \xrightarrow{\eta\circ(\,\cdot\,)} \mathrm{map}_\varphi(X, \langle\Lambda V_K\rangle) \xleftarrow{\mu} \mathrm{map}^\varphi(\Lambda V_K, \Omega(X)) \leftarrow \mathrm{map}^\psi(\Lambda V_K, \mathsf{A}(X)). \tag{3.3}$$

Note furthermore that there are cdga homomorphisms $(H(S^n), 0) \to \Omega(S^n)$ that induces the identity on cohomology, and all of them are homotopic to each other. Together with the combinatorial homotopy group description of Kan sets, we can prove the next result.

Corollary 3.10. *Let f and φ as above. Let $F\colon X \to K$ an extension of f and Ψ be a model for $\eta \circ F$. Then the zig-zag (3.3) together with any quasi-isomorphism $(H(S^n), 0) \to \Omega(S^n)$ induces a homomorphism*

$$\pi_n(\mathrm{map}_f(X, K), F) \to \pi_n(\mathrm{map}^\psi(\Lambda V_K, \mathsf{A}(X)), \Psi) \cong \pi_0(\mathrm{map}^{(\mathrm{id}\otimes\psi, 1\otimes\Psi)}(\Lambda V_K, H(S^n) \otimes \mathsf{A}(X))),$$

with the cdga homomorphism $(\mathrm{id}\otimes\psi, 1 \otimes \Psi)\colon \Lambda V_K \to (H(S^n) \otimes \mathsf{A}(A)) \oplus_{\mathbb{K}\otimes\mathsf{A}(A)} \mathbb{K} \otimes \mathsf{A}(X)$.

We would like to remark that $\pi_0(\mathrm{map}^{(\mathrm{id}\otimes\psi,1\otimes\Psi)}(\Lambda V_K, H(S^n) \otimes \mathsf{A}(X)))$ a priori does not carry a group structure, but it is equipped with the group structure from $\pi_n(\mathrm{map}^\psi(\Lambda V_K, \mathsf{A}(X)), \Psi)$ under the bijection of Corollary 3.10. It is this groups we would like to understand in terms of derivation.

To this end, for any cdga B, we denote by sB the *shifted* chain complex with $(sB)^n = B^{n+1}$ and the 'same' differential, i.e. $d^{sB} = d$. The cdga $H(S^n) \otimes B$ decomposes additively into $H^0(S^n) \otimes B \oplus H^n(S^n) \otimes s^n B$. Therefore, each grading-preserving linear $F\colon C \to H(S^n) \otimes B$ with a chain complex C as domain decomposes uniquely as follows:

$$F = 1 \otimes \theta^0_F + \mathrm{vol}_{S^n} \otimes \theta^n_F. \tag{3.4}$$

An elementary calculation shows that $f\colon C \to H(S^n) \otimes B$ is a homomorphism of graded algebras if and only if θ^0_f is a cdga homomorphism and θ^n_f is a θ^0_f-derivation in the following sense.

Definition 3.11. Let $\Phi\colon C \to B$ be a homomorphism of graded algebras and let $I \subseteq B$ be a differential ideal. A Φ-*derivation* of degree n is a linear map $\theta\colon C \to I$ lowering the degree by n and satisfying

$$\theta(xy) = \theta(x)\Phi(y) + (-1)^{n|x|}\Phi(x)\theta(y)$$

Denote by $\mathrm{Der}^{\Phi}_n(C,I)$ the set of all Φ-derivations of degree n. All together form the graded vector space

$$\mathrm{Der}^{\Phi}(C,I) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Der}^{\Phi}_n(C,I).$$

If Φ is a dga homomorphism between differential graded algebras C and B, then there is a differential $\delta\colon \mathrm{Der}^{\Phi}_n(C,I) \to \mathrm{Der}^{\Phi}_{n-1}(C,I)$ given by $\delta(\theta) = d\theta - (-1)^n\theta d$.

Lemma 3.12. *Let C be a cofibrant cdga, $\Phi\colon C \to B$ a dga homomorphism, $p\colon B \twoheadrightarrow T$ a surjective dga-homomorphism and set $\varphi = p\circ\Phi$. Then, for all $n \geq 1$, there is a natural bijection*

$$\pi_0\big(\mathrm{map}^{(\mathrm{id}\otimes\varphi,1\otimes\Phi)}(C, H(S^n)\otimes B)\big) \xrightarrow{\cong} H_n(\mathrm{Der}^{\Phi}(C,\ker p),\delta); \qquad [F]\mapsto[\theta^n_F] \tag{3.5}$$

Proof. An explicit computation shows that each graded linear map $F\colon C \to H(S^n)\otimes B$, which decomposes uniquely into $1\otimes\theta^0_F + \mathrm{vol}_{S^n}\otimes\theta^n_F$, is an algebra homomorphism if and only if θ^0_F is a graded algebra homomorphism and θ^n_F is a θ^0_F-derivation.

Moreover, the graded linear map F commutes with the differentials if and only if θ^0_F commutes with the differential and θ^n_F is δ-closed, i.e. $\delta(\theta^n_F) = 0$.

By definition, $F \in \mathrm{map}_0^{(\mathrm{id}\otimes\varphi,1\otimes\Phi)}(C, H(S^n)\otimes B))$ is a zero-simplex in this simplicial set, if and only if it is a cdga homomorphism that makes the following diagram commute

$$\begin{tikzcd}
C \arrow[r] \arrow[dr, "{(\mathrm{id}\otimes\varphi,1\otimes\Phi)}"'] & H(S^n)\otimes B \arrow[d, "{(\mathrm{id}\otimes p,\varepsilon\otimes\mathrm{id})}"] \\
& H(S^n)\otimes T \oplus_{\mathbb{K}\otimes T} \mathbb{K}\otimes B,
\end{tikzcd}$$

where $\varepsilon\colon H(S^n) \to H^0(S^n) = \mathbb{K}$ is the augmentation. Since $\varepsilon(\mathrm{vol}_{S^n}) = 0$, we deduce that

$$\theta^0_F = \Phi \qquad \text{and} \qquad p\circ\theta^n_F = 0.$$

To show that the map in the statement is well defined and bijective, we will prove that homotopies are, in fact, in one-to-one correspondence with δ-coboundaries. Abbreviate the cdga $\Lambda[t,dt \mid dt = dt]$ to I. A homotopy H between F_0 and F_1 over T is a dga homomorphism H making the following diagram commute:

$$\begin{tikzcd}
C \arrow[r, "H"] \arrow[dr, "{(1\otimes1\otimes\varphi,1\otimes1\otimes\Phi)}"'] & H(S^n)\otimes I\otimes B \arrow[r, shift left, "\mathrm{ev}_0"] \arrow[r, shift right, "\mathrm{ev}_1"'] \arrow[d] & H(S^n)\otimes B \arrow[d] \\
& H(S^n)\otimes(I\otimes T)\oplus_{\mathbb{K}\otimes I\otimes T}\mathbb{K}\otimes I\otimes B \arrow[r, shift left, "\mathrm{ev}_0"] \arrow[r, shift right, "\mathrm{ev}_1"'] & H(S^n)\otimes T\oplus_{\mathbb{K}\otimes T}\mathbb{K}\otimes B.
\end{tikzcd}$$

The homotopy can be uniquely expressed as

$$\begin{aligned} H(v) =& 1\otimes1\otimes v + 1\otimes t\otimes h^0(v) + \mathrm{vol}_{S^n}\otimes1\otimes\theta^n_{F_0}(v) \\ &+ \mathrm{vol}_{S^n}\otimes t\otimes(\theta^n_{F_1}-\theta^n_{F_0})(v) + \mathrm{vol}_{S^n}\otimes dt\otimes h^n(v). \end{aligned} \tag{3.6}$$

Arguing as before, commutativity of this diagram implies that

(i) $h^0(v) = 0$, which means that H is a pointed homotopy,
(ii) $ph^n(v) = 0$.

From

$$\begin{aligned} dH(v) =& d\Big(1\otimes1\otimes v + 1\otimes t\otimes h^0(v) + \mathrm{vol}_{S^n}\otimes1\otimes\theta^n_{F_0}(v) \\ &\quad + \mathrm{vol}_{S^n}\otimes t\otimes(\theta^n_{F_1}-\theta^n_{F_0})(v) + \mathrm{vol}_{S^n}\otimes dt\otimes h^n(v)\Big). \\ =& 1\otimes1\otimes dv + (-1)^n\mathrm{vol}_{S^n}\otimes1\otimes d\theta^n_{F_0}(v) + (-1)^n\mathrm{vol}_{S^n}\otimes dt\otimes(\theta^n_{F_1}-\theta^n_{F_0})(v) \\ &\quad + (-1)^n\mathrm{vol}_{S^n}\otimes t\otimes d(\theta^n_{F_1}-\theta^n_{F_0})(v) + (-1)^{n+1}\mathrm{vol}_{S^n}\otimes dt\otimes dh^n(v) \end{aligned}$$

and

$$\begin{aligned} H(dv) =& 1 \otimes 1 \otimes dv + \mathrm{vol}_{S^n} \otimes 1 \otimes \theta^n_{F_0}(dv) + \mathrm{vol}_{S^n} \otimes t \otimes \big(\theta_{F_1} - \theta_{F_0}\big)(dv) \\ &+ \mathrm{vol}_{S^n} \otimes dt \otimes h^n(dv), \end{aligned}$$

together with the fact that $\theta^n_{F_j}$ are derivations of degree d that H is a chain map if and only

$$(\theta^n_{F_1} - \theta^n_{F_0})(v) = (-1)^n\big(h^n(dv) - (-1)^{n+1} dh^n(v)\big) = (-1)\big(dh^n(v) - (-1)^{n+1} h^n(dv)\big).$$

From the observation above, we see that $v \mapsto dt \otimes h^n(v)$ is a derivation of degree n, and a straighforward calculation reveals that $v \mapsto h^n(v)$ must be therefore a derivation of degree $n+1$. Thus, $\Theta := h^n$ is a derivation of degree $n+1$ yielding a boundary between θ_{F_0} and θ_{F_1}.

Conversely, a given boundary $\Theta \in \mathrm{Der}^{\Phi}_{n+1}(C, \ker p)$ between $\theta^n_{F_0}$ and $\theta^n_{F_1}$ gives rise to a homotopy over T by plugging in Θ for h^n in equation (3.6). ☐

With all the ingredients at hand, we can now formulate and proof the main theorem of section.

Theorem 3.13. *Let $\iota\colon A \subseteq X$ be the inclusion of a CW-pair with X a finite CW-complex and let Y be a nilpotent topological space. For a given map $f\colon A \to Y$, let $\mathrm{C}_f(X,Y)$ be the topological space of all continuous maps that restrict to f on A and let $F\colon X \to Y$ and extension of f. Let $\mathsf{A}(\iota)\colon \mathsf{A}(X) \twoheadrightarrow \mathsf{A}(A)$ be a surjective model for ι, $(\Lambda V_Y, d)$ a Sullivan model for $\Omega(SY)$, the polynomial differential forms on the singular set SY, and let $\varphi\colon \Lambda V_Y \to \mathsf{A}(A)$ be a rational model for f that is covered by $\Phi\colon \Lambda V_Y \to \mathsf{A}(X)$, the rational model for F.*

Then, for all $n \geq 1$, there is a bijection of sets

$$\pi_n(\mathrm{C}_f(X,Y), F) \otimes \mathbb{K} \longrightarrow H_n(\mathrm{Der}^{\Phi}(\Lambda V_Y, \ker \mathsf{A}(\iota)), \delta).$$

If $n \geq 2$ or if $n = 1$ and $X = Y$ and $F = \mathrm{id}$, then the bijections are also group homomorphisms.

Proof. It is enough to prove the theorem for $\mathbb{K} = \mathbb{Q}$ because $(-) \otimes_{\mathbb{Q}} \mathbb{K}$ is an exact functor and the right-hand side is compatible with tensor products in the sense that $H_n(\mathrm{Der}^{\Phi}(\Lambda V_X, \ker \mathsf{A}(\iota))) \otimes \mathbb{K} \cong H_n(\mathrm{Der}^{\Phi\otimes\mathbb{K}}(\Lambda V_X \otimes \mathbb{K}, \ker[\mathsf{A}(\iota) \otimes \mathbb{K}]))$.

Recall the zig-zag of weak equivalences of relative mapping spaces (3.2) can be further composed the 'homotopy-unit' $\eta\colon K \to \langle \Lambda V_K \rangle$, meaning, the lift of the unit map $K \to \langle \Omega(K) \rangle$ under the quasi-isomorphism $\Lambda V_K \to \Omega(K)$, which gives the zig-zag

$$\mathrm{C}_f(X,Y) \xleftarrow{\simeq} |S\mathrm{C}_f(X,Y)| \xrightarrow{\simeq} |\mathrm{map}_f(S(X), S(Y))| \xrightarrow{\eta\circ(\,\cdot\,)} |\mathrm{map}_{\eta\circ f}(S(X), \langle \Lambda V_Y \rangle))|.$$

By applying homotopy groups and inverting the isomorphisms induced by the left horizontal map together with Corollary 3.10 and Lemma 3.12 we obtain a homomo

$$\begin{aligned} \pi_n(\mathrm{C}_f(X,Y), F) \to \pi_n(\mathrm{map}_{\eta\circ f}(S(X), \langle \Lambda V_Y \rangle), \eta \circ F) &\cong \pi_n(\mathrm{map}^{\eta\circ f}(\Lambda V_Y, \Omega(S(X))), \eta \circ F) \\ &\cong \pi_n(\mathrm{map}^{\varphi}(\Lambda V_Y, \mathsf{A}(X)), \Phi) \end{aligned}$$

where Φ is a model for $\eta \circ F$ and $\varphi = \mathsf{A}(\iota) \circ \Phi$ is a model for $\eta \circ f$.

It is proved in [4, Section 11.2] that the 'homotopy-unit' $\eta\colon K \to \langle \Lambda V_K \rangle$ is a rationalisation if K is a connected, nilpotent, simplicial set of finite $\mathbb{Q}$-type. Since X is a finite nilpotent CW-complex, it follows from the work of Hilton-Mislin-Roitberg [14] that postcomposition with this map yields a $\mathbb{Q}$-localisation $\eta \circ (\,\cdot\,)\colon \mathrm{map}_f(S(X), S(Y)) \to \mathrm{map}_{\eta\circ f}(S(X), \langle \Lambda V_Y \rangle))$. In particular, the above homomorphism $\pi_n(\mathrm{C}_f(X,Y), F) \to \pi_n(\mathrm{map}^{\varphi}(\Lambda V_Y, \mathsf{A}(X)), \Phi)$ is, in fact, an isomorphism after completing the domain with $\mathbb{Q}$.

By Lemma 3.12, we obtain a bijection between $\pi_n(\mathrm{C}_f(X,Y); F) \otimes \mathbb{Q}$ and $H_n(\mathrm{Der}^{\Phi}(\Lambda V_Y, \ker \mathsf{A}(\iota))$.

It remains to show that this bijection is, in fact, a homomorphism if either $n \geq 2$ or $n = 1$ and $\mathrm{C}_f(X,Y) = \mathrm{hAut}_A(X)$ and $F = \mathrm{id}$. We begin with the higher homotopy groups. Under the given hypothesis, the mapping space $\mathrm{C}_f(X,Y)$ is nilpotent and composition with the localisation map

$\ell_{\mathbb{Q}}\colon Y \to Y_{\mathbb{Q}}$ is a rationalisation. By [14, Theorem 2.2] the fibre of a fibration is nilpotent if the total space is. Thus, we have the bijection

$$\begin{aligned}\pi_n(\mathrm{C}_f(X,Y),F)\otimes\mathbb{Q} &= \pi_n(\mathrm{C}_f(X,Y_{\mathbb{Q}}),\ell_{\mathbb{Q}}\circ F) = \pi_n(\mathrm{map}_{\varphi'}(S(X),\langle\Lambda V_Y\rangle,\Phi') \\ &= \pi_n(\mathrm{map}^{\varphi}(\Lambda V_Y,\mathsf{A}(X)),\Phi).\end{aligned}$$

The sum of $[g_1]$ and $[g_2]$ in $\pi_n(\mathrm{C}_f(X,Y),F)$ is represented by the composition

$$S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{g_1\vee g_2} \mathrm{C}_f(X,Y_{\mathbb{Q}})$$

with the first map being the collapse map of course. Its adjoint under the exponential law for mapping spaces yields the composition

$$S^n\times X \xrightarrow{c\times \mathrm{id}_X} (S^n\vee S^n)\times X \xrightarrow{\mathrm{Ad}(g_1\vee g_2)} Y_{\mathbb{Q}}$$

and because of the universal property of localisations, we get a factorisation

$$S^n_{\mathbb{Q}}\times X \xrightarrow{c_{\mathbb{Q}}\times \mathrm{id}} (S^n\vee S^n)_{\mathbb{Q}}\times X_{\mathbb{Q}} \xrightarrow{\mathrm{Ad}(g_1\vee g_2)_{\mathbb{Q}}} Y_{\mathbb{Q}}$$

Since all involved spaces are nilpotent (here we are using $n\geq 2$ to guarantee that $(S^n\vee S^n)_{\mathbb{Q}} = \langle \mathsf{M}_{S^n\vee S^n}\rangle$), the homotopy class of this map has a unique (up to homotopy) algebraic counterpart

$$\begin{tikzcd}
H(S^n)\otimes\mathsf{A}(X) & \mathsf{M}_{S^n\vee S^n}\otimes\Lambda V_X \arrow[l, "c^\bullet\otimes\mathrm{id}"'] \arrow[d, "\simeq"'] & \Lambda V_Y \arrow[l, "(g_1\vee g_2)^\bullet"'] \arrow[dl, "(f\vee g)^\bullet"] \\
(H(S^n)\oplus_{\mathbb{Q}} H(S^n))\otimes\mathsf{A}(X) \arrow[u, "+"] \arrow[r, equal] & H(S^n\vee S^n)\otimes\mathsf{A}(X) \arrow[ul, "H(c)\otimes\mathrm{id}"] &
\end{tikzcd}$$

where $\mathsf{M}_{S^n\vee S^n}$ is the minimal model for $S^n\vee S^n$ and $c^\bullet\colon \mathsf{M}_{S^n\vee S^n}\to H(S^n)$ is a rational model for the collapse map c and $(g_1\vee g_2)^\bullet$ is a rational model for $g_1\vee g_2$. The truncated chain complex $\mathsf{M}^{\leq n}_{S^n\vee S^n}$ is isomorphic to $H(S^n\vee S^n) = H(S^n)\oplus H(S^n)$, so the map $c^\bullet$ agrees with the sum.

On the other, the dga homomorphism $(f\vee g)^\bullet\colon \Lambda V_Y\to (H(S^n)\oplus_{\mathbb{Q}} H(S^n))\otimes\mathsf{A}(X)$ decomposes uniquely into $1\otimes\Phi + \mathrm{vol}_{S^n_1}\otimes\theta^n_f + \mathrm{vol}_{S^n}\otimes\theta^n_g$ from which we deduce

$$\theta^n_{[f]+[g]} = \theta^n_{(f\vee g)\circ c} = \theta^n_f + \theta^n_g.$$

For (ii) we cannot apply the same strategy because $S^1\vee S^1$ is not nilpotent and therefore the composition $S^1\vee S^1\to\langle\mathsf{M}_{S^1\vee S^1}\rangle$ does not need to yield a bijection $[S^1\vee S^1, Y_{\mathbb{Q}}]\cong[\langle\mathsf{M}_{S^1\vee S^1}\rangle, Y_{\mathbb{Q}}]$ and this is the reason why we restrict our consideration to the case $X=Y$ and $F=\mathrm{id}$. Indeed, in this case, $\mathrm{hAut}_A(X)$ carries a monoid structure given by composition. We first replace $\mathsf{A}(\iota)\colon \mathsf{A}(X)\twoheadrightarrow\mathsf{A}(A)$ by a strict Sullivan model $\iota^\bullet\colon \Lambda V_X\twoheadrightarrow\Lambda V_A$ in the sense that the follwing diagram, in which the horizontal arrows are quasi-isomorphisms, (strictly) commute

$$\begin{tikzcd}
\Lambda V_X \arrow[r, "\simeq"] \arrow[d, two heads, "\iota^\bullet"'] & \mathsf{A}(X) \arrow[d, two heads, "\mathsf{A}(\iota)"] \\
\Lambda V_A \arrow[r, "\simeq"] & \mathsf{A}(A).
\end{tikzcd}$$

If ψ and Ψ denote a homotopy lift of φ and Φ, respectively, then the strict model induces an isomorphism $\pi_n(\mathrm{map}^{\psi}(\Lambda V_Y,\Lambda V_X),\Psi)\xrightarrow{\cong}\pi_n(\mathrm{map}^{\varphi}(\Lambda V_Y,\mathsf{A}(X)),\Phi)$ by Lemma 3.7. Thus, $H_n(\mathrm{Der}^{\Psi}(\Lambda V_Y,\ker\iota^\bullet),\delta)$ and $H_n(\mathrm{Der}^{\Psi}(\Lambda V_Y,\ker\iota^\bullet),\delta)$ are isomorphic if they are equipped with the groups structure coming from the homotopy groups.

Theorem 3.2 implies that the groups remain isomorphic under pre-composition with quasi-isomorphisms between Sullivan models $\Lambda V_Y\to\Lambda W_Y$ of Y, so we are allowed to pick the same Sullivan model in domain and target.

We will show that the group structure on $\pi_1(\mathrm{hAut}_A(X), \mathrm{id})$ that comes from the monoid structure agrees with the addition on $\mathrm{Der}^{\Psi}_1(\Lambda V_Y, \ker \iota^\bullet)$. This will prove the second statement, because this groups structure agrees with the fundamental group structure by the Eckmann-Hilton argument.

Under the exponential law, the composition of two maps $g_1, g_2 \colon S^1 \to \mathrm{hAut}_A(X)$ corresponds to the composition of the maps in the upper horizontal line of the following diagram:

$$\begin{tikzcd}
S^1 \times X \arrow[r, "\Delta \times \mathrm{id}_X"] & S^1 \times S^1 \times X \arrow[r, "\mathrm{id}_{S^1} \times \mathrm{Ad}(g_2)"] & S^1 \times X \arrow[r, "\mathrm{Ad}(g_1)"] & X \\
S^1 \times A \arrow[r, "\Delta \times \mathrm{id}_A"] \arrow[u, hook] & S^1 \times S^1 \times A \arrow[r, "\mathrm{id}_{S^1} \times \mathrm{pr}_A"] \arrow[u, hook] & S^1 \times A \arrow[r, "\mathrm{pr}_A"] \arrow[u, hook] & A \arrow[u, hook]
\end{tikzcd}$$

Since all involved spaces are nilpotent, each continuous map in this composition can be modelled by a dga-homomorphism

$$H^1(S^1) \otimes \Lambda V_X \xleftarrow[=\cup \otimes \mathrm{id}]{H(\Delta) \otimes \mathrm{id}} H^1(S^1) \otimes H^1(S^1) \otimes \Lambda V_X \xleftarrow{\mathrm{id} \otimes \mathrm{Ad}(g_2)} H(S^1) \otimes \Lambda V_X \xleftarrow{\mathrm{Ad}(g_1)} \Lambda V_X.$$

The natural decomposition (3.4) together with the fact that $\theta^0_{g_1} = \theta^0_{g_2} = \mathrm{id}$ (because $F = \mathrm{id}$) implies

$$\mathrm{Ad}(g_2) \circ \mathrm{Ad}(g_1) = 1 \otimes 1 \otimes \mathrm{id}_{\Lambda V_X} + \mathrm{vol} \otimes 1 \otimes \theta^1_{g_1} + 1 \otimes \mathrm{vol} \otimes \theta^1_{g_2} + \mathrm{vol} \otimes \mathrm{vol} \otimes \theta^1_{g_2} \otimes^1_{g_1} .$$

From $\mathrm{vol} \cup \mathrm{vol} = 0 \in H^2(S^1)$, we conclude that the composition with $H(\Delta) \otimes \mathrm{id}_{\Lambda V_X}$ yields

$$(H(\Delta) \otimes \mathrm{id}) \circ \mathrm{Ad}(g_2) \circ \mathrm{Ad}(g_1) = 1 \otimes \mathrm{id} + \mathrm{vol} \otimes (\theta^1_{g_1} + \theta^1_{g_2}).$$

In conclusion, the group structure on $\pi_1(\mathrm{hAut}_A(X))$ corresponds under the bijection of Corollary 3.10 to the addition on $H_1(\mathrm{Der}^{\mathrm{id}}(\Lambda V_X, \ker \iota^\bullet), \delta)$ as claimed. ☐

4. Real Homotopy Theory of Orbifold Resolutions

This section is devoted to the proof of the main topological result of this article, namely Theorem C, which essentially says that $\pi_2(B\mathrm{Diff}(M))$ contains a free abelian subgroup if M is obtained from an orbifold by blowing-up singularities.

4.1. Topological Properties of Blow-Up Families.

We begin introducing the twisted families of blow-ups that we use to resolve the singularities with. Recall that the disc bundles associated to the complex line bundle $\mathcal{O}(k) \to \mathbb{C}P^1$ can be obtained from the Hopf-fibration using the Borel construction:

$$D\mathcal{O}(k) = S^3 \times_{S^1, (\,\cdot\,)^k} D^2 = (S^3 \times D^2)/\sim \qquad (p, \lambda) \sim (\mathrm{e}^{\mathrm{i}\theta} p, \mathrm{e}^{\mathrm{i}k\theta} \lambda).$$

We can use the Hopf-fibration once more to construct a $D\mathcal{O}(k)$-bundle over S^2 by setting

$$\mathcal{D}\mathcal{O}(k) = S^3 \times_{S^1, (\,\cdot\,)^1} D\mathcal{O}(k) = (S^3 \times D\mathcal{O}(k))/\sim \qquad (q, [p, \lambda]) \sim (\mathrm{e}^{\mathrm{i}\tau} q, [\mathrm{e}^{\mathrm{i}\tau} p, \lambda]).$$

Since $\mathcal{D}\mathcal{O}(k)$ is obtained from the Borel construction of the Hopf-fibration, the classifying map of this fibre bundle is given by the composition

$$f_{\mathcal{D}\mathcal{O}(k)} \colon S^2 \to BS^1 = \mathbb{C}P^\infty \xrightarrow{B\mathsf{tw}} B\mathrm{Diff}(D\mathcal{O}(k))$$

where $\mathsf{tw} \colon S^1 \to \mathrm{Diff}(D\mathcal{O}(k))$ is the obvious inclusion homomorphism

$$\mathsf{tw} \colon S^1 \to \mathrm{Diff}(D\mathcal{O}(k)), \qquad \mathrm{e}^{\mathrm{i}\tau} \mapsto ([p, \lambda] \mapsto [\mathrm{e}^{\mathrm{i}\tau} p, \lambda]).$$

We would like to compute the order of the elements $[f_{\mathcal{D}\mathcal{O}(k)}] \in \pi_2(B\mathrm{Diff}(M))$.

Proposition 4.1.

1.) The classifying map $f_{\mathcal{D}\mathcal{O}(k)}$ lifts to a map $S^2 \to B\mathrm{Diff}_\partial(D\mathcal{O}(k))$ if $1 \leq k \leq 2$.

2.) The image of $[f_{\mathcal{D}\mathcal{O}(k)}]$ in $\pi_2(B\mathrm{hAut}(D\mathcal{O}(k), S^3/\mathbb{Z}_k))$ has infinite order.

Proof. We start with the proof of the first statement, so let us assume that $k \in \{\pm 1, \pm 2\}$. Since $\mathcal{D}\mathcal{O}(k) \to S^2$ is a fibre bundle over a sphere, it can be obtained by a clutching construction, which means it isomorphic as a fibre bundle to the following pushout

$$\begin{tikzcd}
S^1 \times \partial D\mathcal{O}(k) \arrow[r, "\mathsf{tw}"] \arrow[d, hook] & \partial D\mathcal{O}(k) \arrow[d] \\
D^2 \times D\mathcal{O}(k) \arrow[r] & \mathcal{D}\mathcal{O}(k).
\end{tikzcd}$$

Thus, it suffices to show that the map $\mathsf{tw}\colon S^1 \to \mathrm{Diff}(D\mathcal{O}(k))$ can deformed to a map with values in $\mathrm{Diff}_\partial(D\mathcal{O}(k))$.

To this end, we consider the following diagram this composition fits into the following commutative diagram

$$\begin{tikzcd}
\Omega\mathbb{C}P^1 \arrow[r, "\simeq"] \arrow[d, dashed] & \mathrm{hofib}_1(\mathrm{incl}) \arrow[r] \arrow[d] \arrow[dl, dotted, "\Phi"'] & S^1 \arrow[r] \arrow[d, "\mathsf{tw}"] & S^3 \arrow[d, "\ell"] \\
\mathrm{Diff}_\partial(D\mathcal{O}(k)) \arrow[r, "\simeq"'] \arrow[d] & \mathrm{hofib}_{\mathrm{id}}(\upharpoonleft_{S^3/\mathbb{Z}_k}) \arrow[r] \arrow[d] & \mathrm{Diff}(D\mathcal{O}(k), S^3/\mathbb{Z}_k) \arrow[r, "\upharpoonleft_{S^3/\mathbb{Z}_k}"] & \mathrm{Diff}(S^3/\mathbb{Z}_k) \arrow[d] \\
\mathrm{hAut}_\partial(D\mathcal{O}(k)) \arrow[r, "\simeq"'] & \mathrm{hofib}_{\mathrm{id}}(\upharpoonleft_{S^3/\mathbb{Z}_k}) \arrow[r] & \mathrm{hAut}(D\mathcal{O}(k), S^3/\mathbb{Z}_k) \arrow[r, "\upharpoonleft_{S^3/\mathbb{Z}_k}"] & \mathrm{hAut}(S^3/\mathbb{Z}_k),
\end{tikzcd}$$

where $\ell\colon S^3 \curvearrowright S^3/\mathbb{Z}_k$ is action given by left multiplication[5] and $\mathrm{hAut}(X, A) \subseteq \mathrm{hAut}(X)$ denotes the set of all homotopy equivalences of pairs. The dotted arrow is the map Φ defined as follows: If we use the concrete description $\mathrm{hofib}_1(\mathrm{incl}) = \{(\mathrm{e}^{\mathrm{i}\tau}, \gamma) \in S^1 \times \mathrm{C}([0,1], S^3) \,:\, \gamma(0) = \mathrm{e}^{\mathrm{i}\tau}, \gamma(1) = 1\}$, then

$$\Phi(e^{\mathrm{i}\tau}, \gamma)\colon D\mathcal{O}(k) \ni [p, \lambda] \mapsto \begin{cases} [e^{\mathrm{i}\tau} p, 0], & \text{if } \lambda = 0, \\ [\gamma(|\lambda|) \cdot |\lambda| \cdot (\bar{\lambda}/|\lambda|)^{1/k} \cdot p, 1], & \text{if } \lambda \neq 0. \end{cases}$$

A choice of a null-homotopy $H\colon [0,1] \times S^1 \to S^3$ with $H(\,\cdot\,, 0) = \mathrm{incl}$ and $H(\,\cdot\,, 1) = 1$ defines a map $S^1 \to \mathrm{hofib}_1(\mathrm{incl})$, which is the desired lift.

To prove the second statement, we first restrict to the special case where $k = \pm 1$. We wish to show

$$\pi_1(S^1, 1) \otimes \mathbb{R} \xrightarrow{\pi_1(\mathsf{tw} \otimes \mathbb{R})} \pi_1(\mathrm{hAut}(D\mathcal{O}(\pm 1), S^3), \mathrm{id}) \otimes \mathbb{R}.$$

is injective. To this end, observe that the map tw can be extended by (the adjoint of) the group action of $\mathrm{U}(2)$ on $D\mathcal{O}(\pm 1)$. Together with the tautological action of $\mathrm{PU}(3)$ on $\mathbb{C}P^2$ they fit into the following commutative diagram

$$\begin{tikzcd}
S^1 \times D\mathcal{O}(\pm 1) \arrow[r] \arrow[d, "\mathsf{tw}"'] & \mathrm{U}(2) \times D\mathcal{O}(\pm 1) \arrow[r] \arrow[d] & D\mathcal{O}(\pm 1) \arrow[d] \\
S^1 \times \mathbb{C}P^2 \arrow[r] & \mathrm{PU}(3) \times \mathbb{C}P^2 \arrow[r] & \mathbb{C}P^2,
\end{tikzcd}$$

where the inclusion $S^1 \cdot \mathrm{SU}(2) = \mathrm{U}(2) \to \mathrm{PU}(3)$ is given by $A \mapsto \mathrm{diag}(A, 1)$ and the map $D\mathcal{O}(1) \to D\mathcal{O}(1)/S^3 = \mathrm{Th}(\mathcal{O}(1)) = \mathbb{C}P^2$ is the collapse map. For $D\mathcal{O}(-1)$, we use the map $D\mathcal{O}(-1) \xrightarrow{\mathrm{id} \times \bar{\cdot}} D\mathcal{O}(1) \to \mathbb{C}P^2$ instead.

---

[5] It is here, where we use the assumption $1 \leq k \leq 2$ because only in this case is the normaliser subgroup of $\mathbb{Z}_k$ inside S^3 the group S^3.

Overall, we now end up with the following commutative diagram

$$\begin{tikzcd}
& \Omega\mathbb{C}P^2 \arrow[rr, equal] \arrow[d] & & \Omega\mathbb{C}P^2 \arrow[d] \\
S^1 \arrow[r] & \mathrm{U}(2) \arrow[r] \arrow[d] & \mathrm{hAut}(D\mathcal{O}(1), S^3) \arrow[r, "(\,\cdot\,)/S^3"] & \mathrm{hAut}_*(\mathbb{C}P^2) \arrow[d] \\
& \mathrm{PU}(3) \arrow[rr] & & \mathrm{hAut}(\mathbb{C}P^2),
\end{tikzcd}$$

By a result of Sasao [19], the map $\mathrm{PU}(3) \to \mathrm{hAut}(\mathbb{C}P^2)$ is a real homotopy equivalence. It follows that comparison map $\mathrm{U}(2) \to \mathrm{hAut}_*(\mathbb{C}P^2)$ is a real homotopy equivalence, which implies that $0 \neq [\mathsf{tw}] \in \pi_1(\mathrm{hAut}(D\mathcal{O}(\pm 1), S^3)) \otimes \mathbb{R}$ as claimed.

To generalise this result to all disc-bundles $D\mathcal{O}(k)$ (excluding $k = 0$), we observe that we have a $\mathbb{Z}_k$-action on $D\mathcal{O}(\pm 1)$ that is given by $\zeta_\bullet[p, \lambda] = [p, \zeta \cdot \lambda]$, which agrees with the restriction of the twist action $\mathsf{tw}\colon S^1 \curvearrowright D\mathcal{O}(\pm 1)$. Like the twist action it can be extended to an action on $\mathrm{Th}(D\mathcal{O}(\pm 1)) = \mathbb{C}P^2$ so that $\mathrm{Th}(\mathcal{O}(\pm 1))/\mathbb{Z}_k = \mathrm{Th}(\mathcal{O}(k))$. The quotient map $p_k\colon \mathrm{Th}(\mathcal{O}(\pm 1)) \to \mathrm{Th}(\mathcal{O}(k)) = \mathrm{Th}(\mathcal{O}(\pm 1))/\mathbb{Z}_k$ induces an isomorphism between all real homotopy groups because the involved space are simply connected and quotient maps of finite group actions induce isomorphisms on real homology [5, Theorem III.2.4]. Thus, the (extended) twist actions fit into a commutative diagram

$$\begin{tikzcd}
S^1 \times \mathrm{Th}(\mathcal{O}(\pm 1)) \arrow[r, "\mathrm{tw}/S^3"] \arrow[d] & \mathrm{Th}(\mathcal{O}(\pm 1)) \arrow[d] \\
S^1 \times \mathrm{Th}(\mathcal{O}(k) \arrow[r, "\mathsf{tw}/(S^3/\mathbb{Z}_k)"] & \mathrm{Th}(\mathcal{O}(k))
\end{tikzcd}$$

with the vertical arrows inducing isomorphisms on real homotopy groups. Hence, if the (adjoint of the) lower horizontal map $\mathrm{tw}/(S^3/\mathbb{Z}_k)\colon (S^1, \{1\}) \to (\mathrm{hAut}_*(\mathrm{Th}(\mathcal{O}(k))), \mathrm{id}) \otimes \mathbb{R}$ would present the zero element in $\pi_1(\mathrm{hAut}(\mathrm{Th}(\mathcal{O}(k))), \mathrm{id}) \otimes \mathbb{R}$, then tw/S^3 would also represent the zero element, and we showed above that this is not the case. ☐

Abusing notation, we denote the lift of the twist map given in the proof of the first part of Proposition again by $\mathsf{tw}\colon S^1 \to \mathrm{Diff}_\partial(D\mathcal{O}(k))$ as well as its adjoint map $\mathsf{tw}\colon S^1 \times D\mathcal{O}(k) \to D\mathcal{O}(k)$ if $1 \leq k \leq 2$. This is legitimate because a null-homotopy $H\colon S^1 \times [0,1] \to S^3$ of the inclusion $S^1 \hookrightarrow S^3$ is unique up to homotopy.

From Example 2.13 we immediately obtain the next consequence.

Corollary 4.2. *If $1 \leq k \leq 2$, then there is a $\xi \neq 0 \in \mathbb{R}$ such that*

$$[\theta^1_{\mathsf{tw}}] = \xi \cdot [a_2 \otimes b_3^\vee] \in H_1(\mathrm{Der}(\mathsf{M}_{D\mathcal{O}(\pm 1)}, \ker \mathsf{M}(\iota)), \delta) \cong \pi_1(\mathrm{hAut}_\partial(D\mathcal{O}(\pm 1)), \mathrm{id}) \otimes \mathbb{R}.$$

4.2. **Family Resolutions of tailor-made orbifolds.** With the bundles $\mathscr{D}\mathcal{O}(k) \to S^2$ at our disposal, we are now able to resolve singularities of orbifolds with tubular neighbourhoods of the form $S \times D^4/\mathbb{Z}_2$ in a ‘twisted fashion’ by replacing them with $S \times \mathscr{D}\mathcal{O}(k)$ to obtain fibre bundles over spheres. In this subsection, we will classify this construction through continuous maps between the classifying space $B\mathrm{Diff}_\partial(D\mathcal{O}(k)) \to B\mathrm{Diff}(M)$ and study their effects on rational homotopy groups through appropriate rational models.

From now on, we will only consider orbifolds that are smooth, closed and simply connected. Recall that call a pair $(X, \mathcal{S})$ consisting of an orbifold X and closed smooth manifold $\mathcal{S} \subseteq X$ a *tailor-made orbifold* if $\mathcal{S}$ satisfies the following properties:

(i) Each path component $S \subseteq \mathcal{S}$ has a tubular neighbourhood $\mathrm{Tub}(S)$ of the forms $S \times D^4/\mathbb{Z}_{k_S}$ with $k_S \in \{1, 2\}$ with $\mathbb{Z}_2 \curvearrowright D^4$ acts by reflection at the origin.

(ii) $\mathcal{S}$ contains all singular points of X.

Although one can think of $\mathcal{S}$ as the singular set of the orbifold, we wish to remark that $\mathcal{S}$ is allowed to contain regular points as well.

Since $\mathcal{S}$ is closed, it has only finitely path components, so we may and do assume that the tubular neighbourhoods of different path components do not intersect. We denote the union of these tubular neighborhoods by $\mathrm{Tub}(\mathcal{S})$. Furthermore, since each path component has a trivial normal bundle, $\mathcal{S}$ is an orientable manifold.

Condition (ii) implies that the complement $U = X \setminus \mathrm{Tub}(\mathcal{S})$ is a smooth manifold. Obviously, as a topological space X is the following pushout

$$\begin{tikzcd}
\bigsqcup_{S\in\pi_0(\mathcal{S})} S\times S^3/\mathbb{Z}_{k_S} \arrow[r, equal] & \partial U = \partial\mathrm{Tub}(\mathcal{S}) \arrow[r] \arrow[d] & \mathrm{Tub}(\mathcal{S}) \arrow[r, equal] \arrow[d] & \bigsqcup_{S\in\pi_0(\mathcal{S})} S\times D^4/\mathbb{Z}_{k_S} \\
 & U \arrow[r] & X, &
\end{tikzcd}$$

with $k_S \in \{\pm1, \pm2\}$ of course. A resolving manifold M is now described as the pushout

$$\begin{tikzcd}
\bigsqcup_{S\in\pi_0(\mathcal{S})} S\times S^3/\mathbb{Z}_{k_S} \arrow[r] \arrow[d] & \bigsqcup_{S\in\pi_0(\mathcal{S})} S\times D\mathcal{O}(k_S) \arrow[d] \\
U \arrow[r] & M,
\end{tikzcd}$$

with $k_S \in \{\pm1, \pm2\}$ accordingly to the type of tubular neighbourhood it resolves. By the pushout property, the resolutions maps $\mathsf{bl}\colon D\mathcal{O}(k) \to D^4/\mathbb{Z}_k$ yield a resolution map $\mathsf{bl}\colon M \to X$.

Lemma 4.3. *The resolution map $\mathsf{bl}\colon M \to X$ induces an isomorphism on the fundamental group. In particular, M is simply connected.*

Proof. Of course, we can obtain M from X by resolving one component at a time yielding a finite sequence of orbifolds starting from X and ending with M. Since the resolution maps $\mathsf{bl}\colon \mathcal{O}(k) \to D^4/\mathbb{Z}_k$ induce isomorphisms on the fundamental group, an inductive application of van Kampens theorem implies that the each resolution map in the sequence induces an isomorphism on the fundamental group. Since X is simply connected, so must be M. □

Since the inclusion $\partial U \hookrightarrow U$ induces a surjection on differential forms $\Omega(U) \twoheadrightarrow \Omega(\partial U)$, the pushout description together with Example 2.12 allows us write down a real model of M in terms of its building blocks

$$\begin{tikzcd}
\Omega(\partial U) & \bigoplus_{S\in\pi_0(\mathcal{S})} \mathsf{M}_{S^2}\otimes\Omega(S) \arrow[l] \\
\Omega(U) \arrow[u, two heads] & \mathsf{A}(M) \arrow[l] \arrow[u, two heads]
\end{tikzcd} \tag{4.1}$$

so that

$$\mathsf{A}(M) = \Omega(U) \oplus_{\Omega(\partial U)} \bigoplus_{S\in\pi_0(\mathcal{S})} \mathsf{M}_{S^2}\otimes\Omega(S).$$

Similarly, there is a pushout model for the orbifold given by

$$\mathsf{A}(X) = \Omega(U) \oplus_{\Omega(\partial U)} \bigoplus_{S\in\pi_0(\mathcal{S})} \Lambda V_{D^4}\otimes\Omega(S)$$

with $V_{D^4} = \mathrm{span}\{b_3, \mathrm{vol}_{D^4}\}$ and $db_3 = \mathrm{vol}_{D^4}$. For later purpose, we observe that the two cdgas contain a common differential, graded subalgebra

$$\mathsf{B}(M) = \mathsf{B}(X) = \Omega(U) \oplus_{\Omega(\partial U)} \bigoplus_{S\in\pi_0(\mathcal{S})} \mathbb{R}\cdot 1\otimes\Omega(S).$$

Recall that $\mathsf{M}_{S^2} = \Lambda[a_2, b_3 \,|\, db_3 = a_2^2]$. We index the generators of the summand $\mathsf{M}_{S^2} \otimes 1$ inside $\mathsf{A}(M)$ belonging to S by $a_{2,S}$ and $b_{3,S}$. Observe that system $\{a_{2,S}, b_{3,S} \,:\, S \in \pi_0(\mathcal{S})\}$ is linearly independent in the chain complex $\mathsf{A}(M)/\mathsf{B}(M)$.

Lemma 4.4. *The resolution map* $\mathsf{bl}\colon M \to X$ *induces an injection on real cohomology. Moreover, we have* $H^2(M) = H^2(X) \oplus \operatorname{span}\{a_{2,S} \,:\, S \in \pi_0(\mathcal{S})\}$.

Proof. The resolution map $D\mathcal{O}(k) \to D^4/\mathbb{Z}_k$ restricts to the identity on the boundary, so it is modelled by the dga homomorphism

$$\sigma\colon \Lambda[b_3, \mathrm{vol}_{D^4}] \to \Lambda[a_2, b_3] = \mathsf{M}_{S^2} \quad \text{induced by} \quad b_3 \mapsto b_3/k,\ \mathrm{vol}_D^4 \mapsto a_2^2/k.$$

This dga homomorphism has a retract in the category of chain complexes:

$$\eta\colon \mathsf{M}_{S^2} = \Lambda[a_2, b_3] \to \Lambda[b_3, \mathrm{vol}_{D^4}] \quad \text{given by} \quad a_2^{2n+1} b_3^m \mapsto 0 \ \text{ and } \ a_2^{2n} b_3^m \mapsto k^{n+m} \cdot \mathrm{vol}_4^n b_3^m.$$

Since η is the identity on the subalgebras generated by b_3 it extends to a retract of the real model of $\mathsf{A}(\mathsf{bl}) = \mathrm{id}_{\Omega(U)} \oplus \bigoplus_{S\in\pi_0(\mathcal{S})} \sigma$. (Of course, η and its extension are not dga-homomorphisms), but since cohomology is functorial with respect to chain maps, we obtain a retract of $H(\mathsf{bl})$ on cohomology. Hence $H(\mathsf{bl})$ is injective.

In degree 2, the cokernel of the retract $\mathrm{id}_{\Omega(U)} \oplus \bigoplus_{S\in\pi_0(\mathcal{S})} \eta$ is the direct sum of all $\mathsf{M}^2_{S^2} \otimes 1$, which is spanned by all $a_{2,S}$. Clearly, the elements $(0, a_{2,S} \otimes 1)$ lie in the kernel of the differential of $\mathsf{A}(M)$ and they form a linearly independent system there. For degree reasons, each element in $\mathsf{A}^1(M)$ is the form $(\varphi, 1 \otimes \boldsymbol{\psi})$ with $\varphi \in \Omega^1(U)$ and $\boldsymbol{\psi} = \{\psi_S\}$ is a tuple of differential forms with $\psi_S \in \Omega^1(S)$. Since the differential acts componentwise, we see that the image of $d\colon \mathsf{A}^1(M) \to \mathsf{A}^2(M)$ and the linear hull of $\{(0, a_{2,S} \otimes 1) \,:\, S \in \pi_0(S)\}$ intersect in a trivial fashion. Thus, they are also a linear independent system in cohomology. ☐

Since M is simply connected, we can apply the ‘algorithm’ to compute the minimal model of a simply connected topological space described in [11, p.144 ff], which yields the following result.

Lemma 4.5. *If* $\mathsf{M}_M = \Lambda V_M$ *is the minimal model of* M, *then*

$$\begin{aligned} V_M^2 &= H^2(M) = H^2(X) \oplus \operatorname{span}\{a_{2,S} \,:\, S \in \pi_0(\mathcal{S})\} && d = 0,\\ V_M^3 &= H^3(M) \oplus s\ker\cup\colon \mathsf{S}^2 V_M^2 \to H^4(M), && d\colon s\ker\cup \xrightarrow{\cong} \ker\cup. \end{aligned}$$

The model map $\mu\colon \mathsf{M}_M \to \mathsf{A}(M)$ *satisfies*

$$\begin{aligned} \mu(H^2(X)) &\subseteq \mathsf{B}(M) = \Omega(U) \oplus_{\Omega(\partial U)} \bigoplus_{S\in\pi_0(\mathcal{S})} 1 \otimes \Omega(S),\\ \mu(a_{2,S}) &= (0, a_{2,S} \otimes 1). \end{aligned}$$

We now describe the construction of the fibre bundles that will form the linear independent elements in $\pi_2(B\mathrm{Diff}(M))$. For each $S \in \pi_0(\mathcal{S})$, we can resolve the ‘singularity’ S in a twisted fashion using the non-trivial bundle $\mathcal{D}\mathcal{O}(k)$, which yields a fibre bundle $M \to E_S \to S^2$. Formally, it arises out of the orbifold X as the following pushout:

$$\begin{tikzcd} \partial\mathrm{Tub}(\mathcal{S}) \times S^2 \arrow[r] \arrow[d] & \Big(\bigsqcup_{S\neq S'\in\pi_0(\mathcal{S})} S' \times D\mathcal{O}(k_{S'}) \times S^2\Big) \sqcup \mathcal{D}\mathcal{O}(k_S) \arrow[d] \\ U \times S^2 \arrow[r] & E_S. \end{tikzcd} \tag{4.2}$$

By the pushout property, the projection to S^2 induces a projection map $E_S \to S^2$,. Since $\partial\mathcal{D}\mathcal{O}(k) = S^2 \times S^3/\mathbb{Z}_k$, the projection map turns E_S into a fibre bundle with fibre M. The trivialisation of the boundary extends by construction to the ‘interior’ $U \times S^2 \subset E_S$. Thus, the classifying map f_{E_S} of E_S is given by the composition

$$\begin{tikzcd}
f_{E_S}\colon S^2 \arrow[r] & BS^1 \arrow[r, "B\mathsf{tw}"] & B\mathrm{Diff}_\partial(D\mathcal{O}(k_S)) \arrow[r, "B\mathrm{ext}_S"] \arrow[d] & B\mathrm{Diff}(M) \arrow[d] \\
 & & B\mathrm{hAut}_\partial(D\mathcal{O}(k_S)) \arrow[r, "B\mathrm{ext}"] & B\mathrm{hAut}(M),
\end{tikzcd}$$

where $\mathrm{ext}_S(\varphi)$ is the extension of $\mathrm{id}_S \times \varphi$ by the identity from $S \times D\mathcal{O}(k) \subseteq M$ to the rest of M.

To understand which element the maps f_{E_S} represents in $\pi_2(B\mathrm{hAut}(M))$, or equivalently, which elements their loop maps Ωf_{E_S} represent in $\pi_1(\Omega B\mathrm{hAut}(M), \mathrm{id}) \cong \pi_1(\mathrm{hAut}(M), \mathrm{id})$, we consider the following diagram

$$\begin{tikzcd}
S^1 \arrow[r, "\mathsf{tw}"] \arrow[d, "\mathrm{Ad}(\mathrm{id}_{S^2})"'] & \mathrm{hAut}_\partial(D\mathcal{O}(k_S)) \arrow[r, "\mathrm{ext}_S"] \arrow[d, "\simeq"] & \mathrm{hAut}(M) \arrow[d, "\simeq"] \\
\Omega S^2 \arrow[r, "\Omega B\mathsf{tw}"] & \Omega B\mathrm{hAut}_\partial(D\mathcal{O}(k_S)) \arrow[r, "\Omega f_{E_S}"] & \Omega B\mathrm{hAut}(M)
\end{tikzcd}$$

and remember that $\mathrm{Ad}(\mathrm{id}_{S^2})\colon \pi_1(S^1) \to \pi_1(\Omega S^2) = \pi_2(S^2)$ induces an isomorphism. It is therefore enough to understand the pointed homotopy class of the upper composition.

With help the pushout model $\mathsf{A}(M)$ defined in (4.1) and $H(S^1) \otimes \mathsf{A}(M)$ for the domain $S^1 \times M$, it is fairly easy to write down a real model for the adjoint map $\mathrm{ext}_T \circ \mathsf{tw}\colon S^1 \times M \to M$. Indeed, since it is the extension the map $\mathrm{id}_T \times \mathsf{tw}\colon S^1 \times D\mathcal{O}(k_T) \to D\mathcal{O}(k_T)$ by the projection to the second factor, a real model for $\mathrm{ext}_T(\mathsf{tw})$ on the pushout model for M is given by

$$\begin{tikzcd}
\mathsf{A}(M) \arrow[r, equal] \arrow[d] & \Omega(U) \oplus_{\Omega(\partial U)} \bigoplus_{S \in \pi_0(\mathcal{S})} \mathsf{M}_{S^2} \otimes \Omega(S) \arrow[d, "1 \otimes \mathrm{id}_{\Omega(U)} \oplus \bigoplus_S 1 \otimes (\mathrm{id} + \delta_{ST} \theta^1_{\mathsf{tw}}) \otimes \mathrm{id}"] \\
H(S^1) \otimes \mathsf{A}(M) \arrow[r, equal] & H(S^1) \otimes \Omega(U) \oplus_{H(S^1) \otimes \Omega(\partial U)} \bigoplus_{S \in \pi_0(\mathcal{S})} H(S^1) \otimes \mathsf{M}_{S^2} \otimes \Omega(S)
\end{tikzcd} \tag{4.3}$$

from which we can easily read-off that the corresponding derivation is

$$\theta^1_{\mathrm{ext}_T(\mathsf{tw})} = \xi \cdot a_{2,T} \otimes b^\vee_{3,T}, \tag{4.4}$$

where ξ is the real number in (4.2). Observe that $\mathsf{B}(M) = \Omega(U) \oplus_{\Omega(\partial U)} \bigoplus_{S \in \pi_0(\mathcal{S})} 1 \otimes \Omega(S)$ is a differential graded subalgebra, and that $\theta^1_{\mathrm{ext}_T(\mathsf{tw})}$ vanishes on this subalgebra for every $T \in \pi_0(\mathcal{S})$. Observe further that each $\theta^1_{\mathrm{ext}_T(\mathsf{tw})}$ vanished on $\ker d\colon \mathsf{A}^3(M) \to \mathsf{A}^4(M)$. We compose the derivations with the model map from the 3-truncation of its minimal model $\mu\colon \mathrm{P}_3\mathsf{M}_M \to \mathsf{M}_M \to \mathsf{A}(M)$ to make the calculations for manageable, see Example 2.3 for the definition of $\mathrm{P}_3\mathsf{M}_M$.

Recall from the introduction that $\mathsf{N} \subseteq \pi_0(\mathcal{S})$ is the nice subset of path components with a homological partner, that is $S \in \mathsf{N}$ if and only if there is a path components $S' \pi_0(\mathcal{S})$ different from S such that their homology classes generate the same subvector space $\mathbb{R} \cdot [S] = \mathbb{R} \cdot [S'] \subseteq H_{n-4}(X; \mathbb{R})$.

Proposition 4.6. *The set of homology classes of derivations*

$$\{[\theta^1_{\mathrm{ext}_S(\mathsf{tw})} \circ \mu] \;:\; S \in \mathsf{N}\} \subseteq H_1(\mathrm{Der}^\mu(\mathrm{P}_3\mathsf{M}_M, \mathsf{A}(M)), \delta)$$

spans an $|\mathsf{N}|$*-dimensional subvector space in* $H_1(\mathrm{Der}^\mu(\mathrm{P}_3\mathsf{M}_M, \mathsf{A}(M)), \delta)$.

Proof. We will show that if there is a set of coefficients $\{\lambda_S : S \in \mathsf{N}\}$ such that $\sum \lambda_S \theta^1_{\mathrm{ext}_S(\mathsf{tw})} \circ \mu$ lies in the image of δ_2, then all coefficients must vanish. This implies, in particular, that the set $\{[\theta^1_{\mathrm{ext}_S(\mathsf{tw})} \circ \mu] : S \in \mathsf{N}\}$ is a linear independent system in $H_1(\mathrm{Der}^\mu(\mathrm{P}_3\mathsf{M}_M, \mathsf{A}(M)), \delta)$.

From (4.4) we observe that the image of each linear combination of $\{\theta^1_{\mathrm{ext}_S(\mathsf{tw})} : S \in \pi_0(\mathcal{S})\}$ lies in the ideal $\mathcal{I}$ generated by all $a_{2,S}$, more precisely,

$$\mathcal{I} = \Omega(U) \oplus_{\Omega(\partial U)} \bigoplus_{S \in \pi_0(\mathcal{S})} a_{2,S}\mathsf{M}_{S^2} \otimes \Omega(S).$$

Clearly, this ideal intersects the subalgebra $\mathsf{B}(M)$ in a trivial fashion.

Since derivation from a free algebra are uniquely determined by the image of the underlying vector space, we have

$$\begin{aligned}\mathrm{Der}_2(\mathrm{P}_3\mathsf{M}_M, \mathsf{A}(M)) \cong \mathrm{Hom}(H^2(X), \mathsf{B}^0(M)) \oplus \mathrm{Hom}(\mathrm{span}\{a_{2,S} : S \in \pi_0(\mathcal{S}\}, \mathsf{B}^0(M)) \oplus \\ \mathrm{Hom}(H^3(M), \mathsf{B}^1(M)) \oplus \mathrm{Hom}(s \ker \cup, \mathsf{B}^1(M))\end{aligned}$$

because $\mathsf{A}^1(M) = \mathsf{B}^1(M)$ as $\mathsf{M}^1_{S^2} = \{0\}$, and $\mathsf{A}^0(M) = \mathsf{B}^0(M)$ because $\mathsf{M}^0_{S^2} = \mathbb{R} \cdot 1$.

Since $\mathsf{A}^0(M)$ is a subvector space of $\Omega^0(U) \oplus \mathbb{R}^{\pi_0(\mathcal{S})}$, its elements can be represented by tuples (f, v) consisting of functions $f \in \Omega^0(U)$ and $v \in \mathbb{R}^{\pi_0(\mathcal{S})} = \mathrm{Map}(\pi_0(\mathcal{S}), \mathbb{R}))$ satisfying the boundary condition $f|_S = v(s)$. Hence, if we denote by $a^\vee_{2,S}$ denote the functional that contains $H^2(X)$ in its kernel, and that sends $a_{2,S}$ to 1 and all other $a_{2,S'}$ to 0, then $\mathrm{Hom}(\mathrm{span}\{a_{2,S}\}, \mathsf{A}^0(M))$ is spanned by elements of the form $(f, v)a^\vee_{2,S}$.

Now assume that

$$\begin{aligned}\theta^1 \circ \mu = \sum_{S \in \mathsf{N}} \lambda_S \theta^1_{\mathrm{ext}_S(\mathsf{tw})} \circ \mu &= \sum_{j=1}^{3} \delta_2(F_j) + \sum_{T \in \pi_0(\mathcal{S})} \delta_2(f_T, v_T)a^\vee_{2,T}) \\ &= \sum_{j=1}^{3} \delta_2(F_j) + \sum_{T \in \pi_0(\mathcal{S})} (df_T, 0)a^\vee_{2,T} - (f_T, v_T)a^\vee_{2,T} \circ d)\end{aligned} \tag{4.5}$$

where $F_1 \in \mathrm{Hom}(H^2(X), \mathsf{B}^0(M))$, $F_2 \in \mathrm{Hom}(H^3(M), \mathsf{B}^1(M))$, and $F_3 \in \mathrm{Hom}(s \ker \cup, \mathsf{B}^1(M))$. We make the following observations:

(i) $\delta_2(F_j) = 0$ on $\mathrm{span}\{a_{2,S} : S \in \pi_0(\mathcal{S})\}$ because these elements are closed and, by definition, F_j vanishes on this subset.

(ii) The derivation $\theta^1 \circ \mu$ is uniquely determined by its restriction to $\mathrm{P}_3\mathsf{M}^3_M$, the vector space of elements of degree 3, because it vanishes on $\mathrm{P}_3\mathsf{M}^2_M$.

(iii) The restrictions $\delta_2(F_2)|_{\mathrm{P}_3\mathsf{M}^3_M} = dF_2|_{\mathrm{P}_3\mathsf{M}^3_M}$ and $\delta_2(F_3)|_{\mathrm{P}_3\mathsf{M}^3_M} = dF_3|_{\mathrm{P}_3\mathsf{M}^3_M}$ takes values in $\mathsf{B}^1(M)$.

(iv) Under the decomposition

$$\begin{aligned}\mathsf{S}^2(H^2(M)) = \mathsf{S}^2(H^2(X)) \oplus H^2(X) \otimes \mathrm{span}\{a_{2,S} : S \in \pi_0(\mathcal{S})\} \\ \oplus \mathsf{S}^2(\mathrm{span}\{a_{2,S} : S \in \pi_0(\mathcal{S})\})\end{aligned}$$

the derivation F_1 vanishes on the third summand, while the derivation $\theta^1 \circ \mu$ vanishes on the second and third summand.

From these observations, we draw the following conclusions: By plugging $a_{2,T}$ into (4.5), we derive together with observation (i) that $0 = 0 + (df_T, 0)$, which implies that f_T is closed and hence a constant function. Furthermore, $v_T = f_T|_{\partial U}$ is a constant function, too, with values v_T.

Furthermore, since $\theta \circ \mu$ takes values in the ideal $\mathcal{I}$, which intersects the subalgebra $\mathsf{B}(M)$ in a trivial fashion, we conclude together with observation (ii), (iii) that

$$\delta_2(F_2) = \delta_2(F_3) = 0, \quad \text{and} \quad \delta_2(F_1)|_{\mathrm{P}_3\mathsf{M}^3_M} = -F_1 \circ d|_{\mathrm{P}_3\mathsf{M}^3_M}.$$

Thus, equation (4.5) simplifies to

$$\theta^1 \circ \mu = \sum_{S \in \pi_0(\mathcal{S})} \lambda_S \theta^1_{\mathrm{ext}_S(\mathsf{tw})} \circ \mu = \sum_{T \in \pi_0(\mathcal{S})} -v_T(1,1)a^\vee_{2,T} \circ d - F_1 \circ d|_{\mathrm{P}_3\mathsf{M}^3_M}$$

Since $(0, a_{2,S_0}) \cdot (0, a_{2,S_1}) = 0 \in \mathsf{A}(M)$ if $S_0 \neq S_1$, we have $a_{2,S_0} \cdot a_{2,S_1} \in \ker \cup \subseteq \mathrm{P}_3\mathsf{M}^4_M$. Denote the corresponding element in $\mathrm{P}_3\mathsf{M}^3_M$ by $s(a_{2,S_0} \cdot a_{2,S_1}) \in s\ker\cup$. Since μ is a chain map, we deduce that $\mu(s(a_{2,S_0} \cdot a_{2,S_1})) \in \ker d \subseteq \mathsf{A}^3(M)$ on which θ^1 vanishes. Evaluating the simplified equation at $s(a_{2,S_0}a_{2,S_1})$, yields

$$\begin{aligned}
0 = \theta^1(0) = \theta^1 \circ \mu(s(a_{2,S_0} \cdot a_{2,S_1})) &= \sum_{T \in \pi_0(\mathcal{S})} -v_T(1,1)a_2^\vee(a_{2,S_0} \cdot a_{2,S_1}) - F_1(a_{2,S_0}a_{2,S_1}) \\
&= -\sum_{T \in \pi_0(\mathcal{S})} \delta_{TS_0}v_T(1,1)(0, a_{2,S_1}) + \delta_{TS_1}v_T(1,1)(0, a_{2,S_0}) - 0 \\
&= -v_{S_0}(0, a_{2,S_1}) - v_{S_1}(0, a_{2,S_0}),
\end{aligned}$$

where we used observation (iv) to conclude that $F_1(a_{2,S_0}a_{2,S_1}) = 0$. Since $\{(0, a_{2,S}) : S \in \pi_0(\mathcal{S})\} \subseteq \mathsf{A}^2(M)$ are linearly independent, we deduce $v_S = 0$ for all $S \in \pi_0(\mathcal{S})$.

We are left a collection of real numbers λ_S such that

$$\sum_{S \in \mathsf{N}} \lambda_S \theta^1_{\mathrm{ext}_S(\mathsf{tw})} \circ \mu = \delta_2(F_1). \tag{4.6}$$

for some derivation $F_1 \in \mathrm{Hom}(H^2(X), \mathsf{B}^0(M)) \subseteq \mathrm{Der}_2(\mathrm{P}_3\mathsf{M}_M, \mathsf{A}(M))$. We wish to prove that all $\lambda_S = 0$ for all $S \in \mathsf{N}$.

Let $S_0 \neq S_1 \in \mathsf{N}$ be two path components with $\mathbb{R} \cdot [S_0] = \mathbb{R} \cdot [S_1] \in H_{n-4}(X; \mathbb{R})$. By Poincaré-duality for smooth, closed and oriented orbifolds [20], we deduce that the Thom forms of their tubular neighbourhood generate the same cohomology class (up to a non-zero factor) inside $H^4(X, \mathbb{R})$. In the pushout model for X, the Thom form of the tubular neighbourhood is modelled by $(0, \mathrm{vol}_{D^4,S})$. Since the dga-homomorphism $\Lambda V_{D^4} \to \mathsf{M}_{S^2} = \mathsf{M}_{D\mathcal{O}(k)}$ induced by the resolution map $\mathsf{bl}\colon D\mathcal{O}(k) \to D^4/\mathbb{Z}_k$ sends maps vol_{D^4} to a^2/k, the dga homomorphism $\mathsf{A}(X) \to \mathsf{A}(M)$ induced by the resolution map $\mathsf{bl}\colon M \to X$ sends $(0, \mathrm{vol}_{D^4,S})$ to $(0, a^2_{2,S}/k_S)$. We therefore conclude that there exists non-zero coefficients ν_{S_0} and ν_{S_1} such that

$$\nu_{S_0}a_0^2 + \nu_{S_1}a_1^2 \in \ker(\cup\colon \mathsf{S}^2V^2_M \to H^4(M)),$$

so $\mathrm{s}(\nu_{S_0}a^2_{2,S_0} + \nu_{S_1}a^2_{2,S_1}) \in \mathrm{s}\ker\cup \subseteq V^3_M$.

Obviously, the relation $\sim$ on $\pi_0(\mathcal{S})$ that relates two path components S with S' if and only if their homology classes generate the same vector space in $H_{n-4}(X; \mathbb{R})$ is an equivalence relation. By definition, the set N partitions into a disjoint union of equivalence classes that contain at least two elements. The previous discussion implies that each such equivalence class $S_0/\sim$ generates an $(\#S_0 - 1)$-dimensional vector space inside $s\ker\cup$.

Since the model map $\mathrm{P}_3\mathsf{M}_M \xrightarrow{\mu} \mathsf{A}(M)$ is a dga-homomorphism, we necessarily derive from Lemma 4.5 that

$$\begin{aligned}
\mu(\mathrm{s}(\nu_{S_0}a^2_{2,S_0} + \nu_{S_1}a^2_{2,S_1})) = (\varphi_{S_0,S_1}, \nu_{S_0}b_{3,S_0} \otimes 1 + \nu_{S_1}b_{3,S_1} \otimes 1 + 1 \otimes x_{S_0} + 1 \otimes x_{S_1}) \\
+ \sum_{S \in \pi_0(\mathcal{S})} (\varphi_S, a_{2,S} \otimes \psi_S)
\end{aligned}$$

where $\varphi_S, \varphi_{S_0,S_1} \in \Omega^3(U)$, $x_j \in \Omega^3(S_j)$, and $\psi_S \in \Omega^1(S)$ are closed forms.

Plugging $\mathrm{s}(\nu_{S_0}a^2_{2,S_0} + \nu_{S_1}a^2_{2,S_1})$ into the equation (4.6) and using (4.4) together with observation (iv) yields

$$\begin{aligned}
0 = \delta_2(F)(\mathrm{s}(\nu_0a^2_{2,S_0} + \nu_1a^2_{2,S_1})) = -F_2(\nu_0a^2_{2,S_0} + \nu_1a^2_{2,S_1}) = \sum_{S \in \mathsf{N}} \lambda_S \theta^1_{\mathrm{ext}_S(\mathsf{tw})} \circ \mu(\mathrm{s}(\nu_0a^2_{2,S_0} + \nu_1a^2_{2,S_1})) \\
= \sum_{S \in \mathsf{N}} \lambda_S \theta^1_{\mathrm{ext}_S(\mathsf{tw})}(\nu_{S_0}b_{3,S_0} + \nu_{S_1}b_{3,S_1}) = \lambda_{S_0}(0, \nu_{S_0}a_{2,S_0}) + \lambda_{S_1}(0, \nu_{S_1}a_{2,S_1}).
\end{aligned}$$

By linear independence of $\{(0, a_{2,S}) : S \in \pi_0(\mathcal{S})\}$ inside $\mathsf{A}^2(M)$, we deduce that $\lambda_{S_0} = \lambda_{S_1} = 0$. $\square$

Proof of Theorem C. Since the element $[f_{E_S}] \in \mathrm{im}[\pi_2(B\mathrm{Diff}(M)) \to \pi_2(B\mathrm{hAut}(M))] \otimes \mathbb{R}$ corresponds to $[\mathrm{ext}_T(\mathsf{tw})]$ under the isomorphism $\pi_2(B\mathrm{hAut}(M)) \cong \pi_1(\mathrm{hAut}(M), \mathrm{id})$, we deduce from Theorem F (with $X = M$ and $A = \emptyset$ and the real model $\mathsf{A}(M)$ defined in (4.1) together with Proposition 4.6 that the set of all classes $\{[\mathrm{ext}_S(\mathsf{tw})] : S \in \pi_0(\mathcal{S})\}$ generate a subvector space in $H_1(\mathrm{Der}(\mathsf{M}_M, \mathsf{A}(M)), \delta)$. Under the inclusion, $\mathsf{P}_3\mathsf{M}_M \to \mathsf{M}_M$ yields a linear map $H_1(\mathrm{Der}(\mathsf{M}_M, \mathsf{A}(M)), \delta) \to H_1(\mathrm{Der}(\mathsf{P}_3\mathsf{M}_M, \mathsf{A}(M)), \delta)$, which maps the subvector space to an N-dimensional subvector space by Proposition 4.6. ☐

5. APPLICATIONS TO G_2-GEOMETRY

In this section we apply the previously established topological theory to G_2-moduli spaces. In particular, we will give a proof of Theorem B and derive Theorem A from it. The latter theorem will then be applied to it so a sample of G_2-manifolds constructed in [16].

We start with the proof of Theorem B, which is a fairly easy consequence of our topolgical result Theorem C.

Proof of Theorem B. Let T^7/Γ be a simply connected orbifold and let $\mathcal{S}$ be the union of all path components S of the subspace of singular points that have a tubular neighbourhood $\mathrm{Tub}(S)$ that is diffeomorphic to $T^3 \times D^4/\mathbb{Z}_2$. Assume that T^7/Γ there is a Resolution data (R-data) in the sense of Joyce [17, Definition 14.4.1]. We resolve all singularities not belonging to S according to the construction presented in [17, Section 11.4] to obtain a orbifold X in which $\mathcal{S}$ is the remaining singular set[6]. It comes with a comparison map $\pi\colon X \to T^7/\Gamma$. Because we resolved all singular points not having property (i), the closed, smooth orbifold X is tailor-made for our purpose by construction.

We need to prove that the number $\mathrm{N} = |\mathsf{N}|$ of all path components $S \in \pi_0(\mathcal{S})$ does not decrease under this construction. However, the Poincaré-dual of the homology class represented by $T^3 \subset T^3 \times D^4/\mathbb{Z}_2 \subseteq T^7/\Gamma$ is a four form whose (compact) support is lies in the interior of $T^3 \subset T^3 \times D^4/\mathbb{Z}_2$. By assumption, the tubular neighbourhood $T^3 \times D^4/\mathbb{Z}_2$ does not intersect the tubular neighbourhoods of the remaining singularities, so $\pi\colon \pi^{-1}(T^3 \times D^4/\mathbb{Z}_2) \to T^3 \times D^4/\mathbb{Z}_2$ is a homeomorphism (that is smooth away from the 'cone-tip' $T^3 \times \{0\}$). Thus, the Poincaré dual pulls back to a differential forms whose Poincaré dual homology class is represented by $\pi^{-1}(T^3 \times \{0\})$. It follows that two path components $S \cong T^3$ and $S' \cong T^3$ are homologous in T^7/Γ if and only if they are homologous in X, so N remains unchanged ☐

In the formulation of the next results, which is a refined formulation of Theorem A, we use notation from the previous section.

Proof of Theorem A. The underlying smooth manifold M is obtained from T^7/Γ by replacing $S \times D^4/\mathbb{Z}_2$ by $S \times D\mathcal{O}(-2)$ for all $S \in \pi_0(\mathcal{S})$ (and resolving the remaining singularities as explained in [17, Chapter 11]). Following Section 6.1 and Section 6.2 in [7][7], the bundle

$$E_{M,\{S\}} = (M \setminus S \times D\mathcal{O}(-2)) \cup_{S \times \mathbb{R}P^3 \times S^2} \cup S \times \mathcal{D}\mathcal{O}(-2) \to S^2$$

carries a fibre-wise torsion-free G_2-structure.

The underlying bundle $E_{M,\{S\}}$ can be alternatively constructed by resolving all singularity components of T^7/Γ not belonging to $\mathcal{S}$ to obtain a closed, smooth, simply-connected orbifold X that is tailor-made for our purpose. The bundle $E_S \to S^2$ obtained from the construction (4.2) obviously agrees with $E_{M,\{S\}}$. By Theorem B, their classifying maps generate an $|\mathsf{N}|$-dimensional vector space inside $\pi_2(B\mathrm{Diff}(M)_0) \otimes \mathbb{R}$. Since the fibre bundles carry a fibre-wise torsion free G_2-structure, this subgroup lifts to an $|\mathsf{N}|$-dimensional subvectorspace in $\pi_2(\mathcal{G}_2^{\mathrm{tf}}(M) /\!\!/ \mathrm{Diff}(M)_0) \otimes \mathbb{R}$, and

---

[6] Alternatively, carry out Joyce's construction to obtain M and blow down the resolution of the singularities belonging to $\mathcal{S}$ to obtain X

[7] In [7], the spaces $D\mathcal{O}(-2)$ and $\mathcal{D}\mathcal{O}(-2)$ are denoted by $EH_{\leq 1}$ and $\mathcal{E}\mathcal{H}_{\leq 1}$

since the homomorphism $\pi_2(\mathcal{G}_2^{\mathrm{tf}}(M)/\!\!/\mathrm{Diff}(M)_0) \to \pi_2(\mathcal{G}_2^{\mathrm{tf}}(M)/\mathrm{Diff}(M)_0)$ induced by the obvious comparison map is injective, see [7, Theorem B], the claim follows. ☐

We apply this theorem to the examples presented in [16].

Example 5.1. This is the first class of examples of generalised Kummer constructions discussed in Section 3.1 in [16]. More precisely, we are discussing Example 3 and Example 4 in detail while providing a sketch for Example 6 in loc. cit. Example 5 in loc. cit. concerns G_2-manifolds that are not simply connected and thus falls out of our set-up.

The general set-up is the following: For $T^7 = (\mathbb{R}/\mathbb{Z})^7$ with coordindates $x_1, \dots, x_7$, Joyce considers the group $\Gamma \cong \mathbb{Z}_2^3$ generated by the involutions

$$\begin{aligned}
\alpha(x_1, \dots, x_7) &= (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7)\\
\beta(x_1, \dots, x_7) &= (b_1 - x_1, b_2 - x_2, x_3, x_4, -x_5, -x_6, x_7)\\
\gamma(x_1, \dots, x_7) &= (c_1 - x_1, x_2, c_3 - x_3, x_4, -x_5, x_6, -x_7)
\end{aligned}$$

with $b_1, b_2, c_1, c_3, c_5 \in \{0, 1/2\}$. This implies that their fixpoint sets are given by

$$\begin{aligned}
\mathrm{Fix}(\alpha) &= \left\{0, \frac{1}{2}\right\}^4 \times T^3(567),\\
\mathrm{Fix}(\beta) &= \left\{\left(\frac{b_1 + \varepsilon_1}{2}, \frac{b_2 + \varepsilon_2}{2}, \frac{\varepsilon_5}{2}, \frac{\varepsilon_6}{2}\right) \;:\; \varepsilon_j \in \{0, 1\}\right\} \times T^3(347),\\
\mathrm{Fix}(\gamma) &= \left\{\left(\frac{c_1 + \varepsilon_1}{2}, \frac{c_3 + \varepsilon_3}{2}, \frac{c_5 + \varepsilon_5}{2}, \frac{\varepsilon_7}{2}\right) \;:\; \varepsilon_j \in \{0, 1\}\right\} \times T^3(246).
\end{aligned}$$

From this description it is clear any two canonical identifications of $T^3 \cong (y_1, y_2, y_3, y_4) \times T^3(567)$ yield homotopic maps $T^3 \to T^7$. Thus, each two path components in $\mathrm{Fix}(\alpha)$ generate the same homology class in $H_3(T^7; \mathbb{R})$ and so their images agree in $H_3(T^7/\Gamma; \mathbb{R})$. The same is true for the other two group elements.

In the examples Joyce considers, the other group elements of Γ do not have fix points, so the singular set of T^7/Γ is the image of the fix point sets described above under the quotient map $T^7 \to T^7/\Gamma$. It remains to read off the set $\mathcal{S}$ of path components that have a tubular neighbourhood of the form $T^3 \times D^4/\mathbb{Z}_2$ and the nice subset N from Joyce's article and derive the conclusion from Theorem A.

(i) In Example 3 in [16], the set $\mathcal{S}$ consists of the image of $\mathrm{Fix}(\alpha)$, $\mathrm{Fix}(\beta)$, and $\mathrm{Fix}(\gamma)$. From the discussion above, we conclude $\mathsf{N} = \pi_0(\mathcal{S})$, which is a set of 12-elements. Theorem A now implies that $\mathbb{Z}^{12} \subseteq \pi_2(\mathcal{G}_2^{\mathrm{tf}}(M)/\mathrm{Diff}(M)_0)$.

(ii) In Example 4 in [16], the set $\mathcal{S}$ is generated by the image of $\mathrm{Fix}(\alpha)$ and $\mathrm{Fix}(\beta)$, and the discussion above shows that $\mathsf{N} = \pi_0(\mathcal{S})$, which is a set of 8 elements. Theorem A now implies that $\mathbb{Z}^8 \subseteq \pi_2(\mathcal{G}_2^{\mathrm{tf}}(M)/\mathrm{Diff}(M)_0)$.

Example 6 in [16] uses an additional involution

$$\delta(x_1, \dots, x_7) = (1/2 + x_1, x_2, 1/2 + x_3, 1/2 + x_4, 1/2 + x_5, x_6, x_7)$$

and chooses $(b_1, b_2, c_1, c_3, c_5) = (1/2, 0, 1/2, 0, 1/2)$. The only elements in the group $\langle \alpha, \beta, \gamma, \delta\rangle \cong \mathbb{Z}_2^4$ that have fix points are α, β, γ, and $\alpha\beta\delta$ and, according to Joyce, only $\mathrm{Fix}(\gamma)$ and $\mathrm{Fix}(\alpha\beta\delta)$ contribute each to two path components in T^7/Γ which lie in $\mathcal{S}$. The fix point set of $\alpha\beta\delta$ is given by

$$\mathrm{Fix}(\alpha\beta\delta) = \left\{\left(\frac{1/2 + \varepsilon_3}{2}, \frac{1/2 + \varepsilon_4}{2}, \frac{1/2 + \varepsilon_5}{2}, \frac{\varepsilon_6}{2}\right) \;:\; \varepsilon_j \in \{0, 1\}\right\} \times T^3(127)$$

and hence we have $\pi_0(\mathcal{S}) = \mathsf{N}$ with $|\mathsf{N}| = 4$.

Example 5.2. We now discuss the Examples 7-11 in Section 3.2 of [16] except Example 10, which is not simply-connected. The general set-up is the following: Let $\mathbb{R}^7 = \mathbb{C}^3 \times \mathbb{R}$ with standard complex coordinates z_1, z_2, z_3 and real coordinate x. Let $\Lambda \cong \mathbb{Z}^6$ be a lattice and set $T^7 = (\mathbb{C}^3 \times \mathbb{R})/(\Lambda \times \mathbb{R})$. For two complex numbers u, v with $u^a = v^a = 1$ for some natural number a, Joyce considers the isometries α, β defined by

$$\alpha(z_1, z_2, z_3, x) = (uz_1, vz_2, \overline{uv}z_3, x + a^{-1}),$$
$$\beta(z_1, z_2, z_3, x) = (-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -x).$$

Under the assumption that the two isometries α and β preserve $\Lambda \times \mathbb{R}$ they descend to an action of the dihedral $\Gamma = \langle \alpha, \beta \rangle \cong D_{2a}$ on T^7 that preserves the standard G$_2$-structure. Joyce observed that α^j does not have fixpoints if $\alpha^j \neq 1$. Furthermore, if a is odd, then $\mathrm{Fix}(\beta\alpha^j)$ and $\mathrm{Fix}(\beta)$ generated the same singularities and if a is even, then the singularity components in T^7/D_{2a} partition into the ones generated by $\mathrm{Fix}(\beta)$ and $\mathrm{Fix}(\beta\alpha)$.

Joyce proved that if a is odd, then the number of singularity components in T^7/Γ agrees with the number of components of $\mathrm{Fix}(\beta)$. If a is even, then $\mathrm{Fix}(\beta)$ and $\mathrm{Fix}(\beta\alpha)$ both split into two sets of equal size, distinguished by the value of x, which are interchanged by the α.

Independent of the parity of a, the tubular neighbourhoods of all fixpoints are of the form $T^3 \times D^4/\mathbb{Z}_2$, see [16, p.356], which is precisely what we want.

Example 7: Here, $u = v = \mathrm{e}^{2\pi \mathrm{i}/3}$, $a = 3$ the lattice $\Lambda = \mathbb{Z}^3 \oplus \mathrm{e}^{2\pi \mathrm{i}/3}\mathbb{Z}^3$. We have $\mathrm{Fix}(\beta) = \mathbb{R}^3/\mathbb{Z}^3 \times \{0\}^3 \times \{0, 1/2\}$. Clearly, the two tori generate the same homology class in T^7/D_{2a} because the obvious two embeddings are homotopic. Thus $\pi_0(\mathcal{S}) = \mathsf{N}$ and $|\mathsf{N}| = 2$. It follows that $\pi_2(\mathcal{G}_2^{\mathrm{tf}}(M)/\mathrm{Diff}(M)_0) \supseteq \mathbb{Z}^2$.

Example 8: This one is more interesting: $u = v = \mathrm{e}^{\pi \mathrm{i}}$, $a = 6$, and Λ as in Example 7. Now $\mathrm{Fix}(\beta)$ is the same Example 7, but now α^3 identifies the two different component with each other so that they generate a single component in T^7/D_{12}. Likewise the two components

$$\mathrm{Fix}(\beta\alpha) = \{(r_1 \mathrm{e}^{\pi \mathrm{i}/6}, r_2 \mathrm{e}^{\pi \mathrm{i}/6}, r_1 \mathrm{i} \mathrm{e}^{\pi \mathrm{i}/3}, y) \ : y \in \{3/12, 5/12\}\}$$

contributes a single component in T^7/D_{12}. However, the two path components generate different homology classes in $H_3(T^7/D_{2a}; \mathbb{R}) = H_3(T^7; \mathbb{R})^{D_{2a}}$ because the transfer applied to a single component yield elements that are not co-linear over the real numbers. Thus, $\mathsf{N} = \emptyset$ and Theorem A is inconclusive.

Example 9: Here, $u = v = \mathrm{i}$, $a = 4$ and $\Lambda = \mathbb{Z}^3 \oplus \mathrm{i}\mathbb{Z}^3$. Decompose $z_j = s_j + \mathrm{i}t_j$ into real and imaginary part. Joyce calculated the fixpoint sets to be

$$\mathrm{Fix}(\beta) = \{(s_1 + \varepsilon_1 \mathrm{i}, s_2 + \varepsilon_2 \mathrm{i}, s_3 + \varepsilon_3 \mathrm{i}, \varepsilon_4) \ : \ \varepsilon_j \in \{0, 1/2\}\}$$

and

$$\mathrm{Fix}(\beta) = \{(s_1 - s_1 \mathrm{i}, s_2 - s_2 \mathrm{i}, t_3 + \varepsilon_3 + t3\mathrm{i}, \varepsilon_4) \ : \ \varepsilon_3 \in \{0, 1/2\}, x \in \{3/8, 7/8\}\}.$$

Since the involution α^2 acts freely on $\mathrm{Fix}(\beta)$, the 16 components give rise to eight components in T^7/Γ. Moreover, all fixpoint components have isotopic inclusions and therefore generate the same homology class.

Likewise, the four fix point components of $\mathrm{Fix}(\beta\alpha)$ contribute to two components in T^7/Γ generating the same homology class. It follows that $|\mathsf{N}| = \pi_0(\mathcal{S})| = 10$.

Example 11:

Here $u = \mathrm{e}^{\pi \mathrm{i}/3}$, $v = \mathrm{e}^{2\pi \mathrm{i}/3}$, $a = 6$, and Λ the lattice given by

$$\Lambda = (\mathbb{Z} + \mathrm{e}^{2\pi \mathrm{i}/3}\mathbb{Z}) \oplus (\mathbb{Z} + \mathrm{e}^{2\pi \mathrm{i}/3}\mathbb{Z}) \oplus (\mathbb{Z} + \mathrm{i}\mathbb{Z}).$$

Joyce computed the fixpoint sets to be

$$\mathrm{Fix}(\beta) = \{(s_1, s_2, s_3 + \varepsilon_3 \mathrm{i}, \varepsilon_4) \ : \ \varepsilon_j \in \{0, 1/2\}\}$$

while

$$\mathrm{Fix}(\beta\alpha) = \left\{ (\lambda_1 \mathrm{e}^{\pi \mathrm{i}/3}, \lambda_2 \mathrm{e}^{\pi \mathrm{i}/6}, \varepsilon_3 + t_3 \mathrm{i}, \varepsilon_4) \,:\, \lambda_j, t_3 \in \mathbb{R}, \varepsilon_3 \in \{0, 1/2\}, \varepsilon_4 \in \{5/12, 11/12\} \right\}.$$

Thus, $\mathrm{Fix}(\beta)$ and $\mathrm{Fix}(\beta\alpha)$ both contribute two singularity components to T^7/Γ. We conclude $|\mathsf{N}| = |\pi_0(\mathcal{S})| = 4$.

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